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ABSTRACT

This proceedings from the annual conference of the International Group for the Psychology of Mathematics Education includes the following papers: "Street Mathematics and School Mathematics" (Carraher); "A Look at the Affective Side of Mathematics Learning in Hungarian Secondary Schools" (Klein & Habermann); "Beyond Constructivism: Learning Mathematics at School" (Nesher); "Reconstructive Learning" (Streefland); "Perceptions of Teachers' Questioning Styles" (Ainley); "Teacher Change as a Result of Counselling" (Albert, Friedlander, & Fresko); "Codidactic System in the Course of Mathematics" (Alibert); "The Construction of Arithmetic Structures by a Group of Three Children Across Three Tasks" (Alston & Maher); "Career Choice, Gender and Attribution Patterns of Success and Failure in Mathematics" (Amit); "A Classification of Students' Errors in Secondary Level Algebra" (Becker); "Teachers' Written Explanations to Pupils about Algebra" (Bliss & Sakonidis); "Algebra - Choices in Curriculum Design" (Bell); "Number Naming Grammars and the Concept of 10" (Bell); "The Kindergartner's Understanding of Discrete Quantity" (Bergeron & Herscovics); "A Review of Research on Visualisation in Mathematics Education" (Bishop); "Acquisition of Meanings and Evolution of Strategies in Problem Solving from the Age of 7 to the Age of 11" (Boero); "The Relationship between Capacity to Process Information and Levels of Mathematical Learning" (Boulton-Lewis); "Mathematical Vulnerability" (Brandau); "Cognitive Psychology and Mechanistic versus Realistic Arithmetic Education" (van den Brink); "Proof and Measurement: An Unexpected Misconception" (Chazan); "'Discrete' Fraction Concepts and Cognitive Structure" (Clements & Lean); "Algorithmic Thinking of Deaf Pupils" (Cohors-Fresenborg); "The Effect of Order-Coding and Shading of Graphical Instructions on the Speed of Construction of a Three-Dimensional Object" (Cooper); "New Contexts for Learning in Mathematics" (Crawford); "Quelques Developpements Recents des Recherches sur la Discussion Autour de Problemes" (Drouhard, Lymberopoulou-Fioravantes, Nikolakarou, Paquelier); "On Helping Students Construct the Concept of Quantification" (Dubinsky); "Children's Learning in a Transformation Geometry Microworld" (Edwards); "Some Cognitive Preference Styles in Studying Mathematics" (El-Faramawy); "Exploring Children's Perception of Mathematics Through Letters and Problems Written by Children" (Ellerton); "The Attitudes and Practices of Student Teachers of Primary School Mathematics" (Ernest); "Contexts and Performance in Numerical Activity Among Adults" (Evans); "Pre-service Teachers' Conceptions of the Relationships between Functions and Equations" (Even); "An

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Experimental Study of Solving Problems in Addition and Subtraction by First-Graders" (Feiyu & Shanghe); "Beyond Ratio Formula" (Lin); "Eye Fixations During the Reading and Solution of Word Problems Containing Extraneous Information" (Fry); and "The Meaning of 'X' in Linear Equation and Inequality" (Fujii). (MKR)

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2

Volume I

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PROCEEDINGS
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12th International Conference

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PREFACE

The 12th annual conference of the PME is the first meeting in the history of the International Group for the Psychology of Mathematical Education held in an East-European socialist country. The conference takes place in the old episcopal city Veszprém, from July 20th to July 25th, 1988.

There are a number of different ways in which participants at the conference may make a contribution: research reports, poster displays, working groups /initiated in 1984/ and discussion groups /initiated in 1986/. One session is devoted to the preparation for the ICME-6 presentations of the PME. An innovation at this conference is that following each group of papers of similar topics a summary session will be held to discuss and evaluate the achievements in the given territory. The discussion sessions will be held in the following topics:

1. Algebra
2. Rational numbers
3. Early numbers
4. Metacognition
5. Teachers' beliefs
6. Problem solving
7. Computer environments
8. Social factors

We would like to thank Thomas A. Romberg, Claude Comiti, Kathleen Hart, Richard Lesh, Tommy Dreyfus and Colette Laborde for volunteering to chair and introduce these evaluation sessions.

87 research papers have been submitted to the conference. All of them have been evaluated by at least two reviewers and the final decision on the acceptance of the papers has been done at a session of the International Program Committee in Budapest, based on the reports of the reviewers. The members of the International Committee of the PME and the International Program Committee have served as reviewers for the submitted papers.

The order in which the research papers appear in these two volumes is alphabetic /according to the first author of the paper/ except for the invited plenary papers that are taken first. Therefore the order of the papers in the volumes does not necessarily reflect the order of presentation within the meeting itself. Any particular paper can be located by consulting either the table of contents at the beginning or the alphabetical list of contributors at the end. We would like to thank the International Program Committee, the Local Organizing Committee and the reviewers fo their assistance in the preparation of this conference.

International Program Committee:

Chairman: János Surányi /Hungary/
Secretary: László Mérő /Hungary/
Members: András Ambrus /Hungary/
Katalin Bognár /Hungary/
Joop van Dormolen /The Netherlands/
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Local Organizing Committee:

President: Ferenc Genzwein - general director of OOK /Hungary/
Secretaries: Andrea Borbás and Maria Dax /Hungary/

HISTORY AND AIMS OF THE PME GROUP

At eh Second International Congress on Mathematical Education /ICME 2, Exeter, 1972/ Professor E. Fischbein of Tel Aviv University, Israel, instituted a working group bringing together people working in the area of the psychology of mathematics education. At ICME 3 /Karlsruhe, 1976/ this group became one of the two groups affiliated to the International Commission for Mathematical Instruction /ICMI/.

According to its Constitution the major goals of the group are:

1./ to promote international contacts and the exchange of scientific information in the psychology of mathematical education,

2./ to promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers,

3./ to further a deeper and more correct understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

MEMBERSHIP

1./ Membership is open to persons involved in active research in furtherance of the Group's aims, of professionally interested in the results of such research.

2./ Membership is on an annual basis and depends on payment of the subscription for the current year /January to December/

3./ The subscription can be paid together with the conference fee.

The present officers of the group are as follows:

President: Pearla Nesher /Israel/
Vice-President: Willibald Dörfler /Austria/
Secretary: Joop van Dormolen /The Netherlands/
Treasurer: Carolyn Kieran /Canada/

TABLE OF CONTENTS

TABLE OF CONTENTS

VOLUME I Page

I. PLENARY PAPERS

1. Street mathematics and school mathematics
   Terezinha Nunes Carraher /Brasil/ 1

2. A look at the affective side of mathematics learning in Hungarian Secondary schools
   Sándor Klein, Gustav M. Habermann /Hungary/ 24

3. Beyond constructivism /Learning mathematics at school/
   Perla Nesher /Israel/ 54

4. Reconstructive Learning
   Leen Streefland /The Netherlands/ 75

II. RESEARCH PAPERS

1. Perceptions of teachers'questioning styles
   Janet Ainley 92

2. Teacher change as a result of counselling
   Jeanne Albert, Alex Friedlander, Barbara Fresko 1oo

3. Codidactic system in the course of mathematics: how to introduce it?
   Daniel Alibert 1o9

4. The construction of arithmetic structures by a group of three children across three tasks
   Alice Alston, Carolyn A.Maher 117

5. Career choice, gender and attribution patterns of success and failure in mathematics
   Miriam Amit 125

6. A classification of students' errors in secondary level algebra
   Gerhard Becker 131

7. Teachers'written explanations to pupils about algebra
   Joan Bliss, H. Sakonidis 139

8. Algebra - Choices in curriculum design: an exploratory teaching experiment
Alan Bell 147

9. Number naming grammars and the concept of "10"
Garry Bell 154

10. The kindergartners understanding of discrete quantity
Jacques C.Bergeron, Nicolas Herscovics 162

11. A review of research on visualisation in mathematics education
Alan J. Bishop 170

12. Acquisition of meanings and evolution of strategies in problem solving from the age of 7 to the age of 11 in a curricular environment
Paolo Boero 177

13. The relationship between capacity to process information and levels of mathematical learning
Gillian M.Boulton-Lewis 185

14. Mathematical vulnerability
Linda Brandau 193

15. Cognitive psychology and mechanistic versus realistic arithmetic education
Jan van den Brink 201

16. Proof and measurement: an unexpected misconception
Daniel Chazan 207

17. "Discrete" fraction concepts and cognitive structure
M.A./Ken/ Clements, G.A. Lean 215

18. Algorithmic thinking of deaf pupils - report on a three years curriculum project
Elmar Cohors-Fresenborg 223

19. The effect of order-coding and shading of graphical instructions on the speed of construction of a three-dimensional object
Martin Cooper 231

20. New contexts for learning in mathematics
Kathryn Crawford 239

21. Quelques développements récents des recherches sur la discussion autour de problèmes
Jean-Philippe Drouhard, Hélène Lymberopoulou-Fioravantes, Hélène Nikolakarou, Yves Paquelier 247

22. On helping students construct the concept of quantification
Ed Dubinsky 255

23. Children's learning in a transformation geometry microworld
Laurie D. Edwards 263

24. Some cognitive preference styles in studying mathematics
Hamdy El-Faramawy 271

25. Exploring children's perception of mathematics through letters and problems written by children
Nerida F.Ellerton 280

26. The attitudes and practices of student teachers of primary school mathematics
Paul Ernest 288

27. Contexts and performance in numerical activity among adults
Jeffrey T.Evans 296

28. Pre-service teachers' conceptions of the relationships between functions and equations
Ruhama Even 304

29. An experimental study of solving problems in addition and substraction by first-graders
Cao Feiyu, Cai Shanghe 312

30. Beyond ratio formula
Fou-Lai Lin 320

31. Eye fixations during the reading and solution of word problems containing extraneous information: relation to spatial visualization ability
Carol J.Fry 326

32. The meaning of "X" in linear equation and inequality: preliminary survey using cognitive conflict problems
Toshiakira Fujii 334

VOLUME II

III. RESEARCH PAPERS /continued/

33. Intervention in a mathematics course at the college level
Linda Gattuso, Raynald Lacasse 342

34. The education of talented children
Ferenc Genzwein 349

35. The development of a model for competence in mathematical problem solving based on systems of cognitive representation
Gerald A.Goldin 358

36. Pilot work on secondary level
Mária Halmos 366

37. On the textual and the semantic structure of mapping rule and multiplicative compare problems
Guershon Harel, Thomas Post, Merlyn Behr 372

38. Forging the link between practical and formal mathematics
Kethleen M.Hart, A.Sinkinson 380

39. The kindergartners' understanding of the notion of rank
Nicolas Herscovics, Jacques D.Bergeron 385

40. Initial research into the understanding of percentages
Rina Hershkowitz, Tirza Halevi 393

41. Structuring and destructuring a solution: an example of problem solving work with the computer
J.Hillel, J.-L. Gurtner, C.Kieran 402

42. Metacognition: The role of the "Inner teacher"
Ichiei Hirabayashi, Keiichi Shigematsu 410

43. Formalising intuitive descriptions in a parallelogram LOGO microworld
Celia Hoyles, Richard Noss 417

44. One mathematics teacher
Barbara Jaworski 425

45. Learning the structure of algebraic expressions and equations
Carolyn Kieran 433

46. The influence of teaching on Children's strategies for solving proportional and inversely proportional word problems
Wilfried Kurth 441

47. Constructing bridges from intrinsic to cartesian geometry
Chronis Kynigos 449

48. Concepts in secondary mathematics in Botswana
Hilda Lea 457

49. A developmental model of a first level competency in procedural thinking in LOGO: "May be we're not expert, but we're competent"
Tamara Lemerise 463

50. The naive concept of sets in elementary teachers
Liora Linchevski, Shlomo Vinner 471

51. Concrete introduction to programming languages and observation of Piagetian Stages
F. Lowenthal 479

52. Cognitive and metacognitive shifts
John H.Mason, P.Joy Davis 487

53. Learning mathematics cooperatively with CAI
Zemira Mevarech 495

54. Mathematical pattern-finding in elementary school, focus on pupils' strategies and difficulties in problem-solving
Nobuhiko Nohda 503

55. The construction of an algebraic concept through conflict
Alwyn Olivier 511

56. Gender and mathematics: the prediction of choise and achievement
Wilma Otten, Hans Kuyper 519

57. Teaching and learning methods for problem solving: some theoretical issues and psychological hypotheses
Janine Rogalski, Aline Robert 528

58. Student-sensitive teaching at the tertiary level: a case-study
Pat Rogers 536

59. Strategy choice in solving additions: memory of understanding of numerical relations
Analucia Dias Schliemann 544

60. Representation of functions and analogies
Baruch Schwarz, Maxim Bruckheimer 552

61. Operational vs. structural method of teaching mathematics-case study
Anna Sfard 560

62. Epistemological remarks on functions
Anna Sierpinska 568

63. Formative evaluation of a constructivist mathematics teacher inservice program
Martin A.Simon 576

64. Construction and reconstruction: the reflective practice in mathematics education
Beth Southwell 584

65. Graphical lesson patterns and the process of knowledge development in the mathematics classroom
Heinz Steinbring, Rainer Bromme 593

66. Longer term conceptual benefits from using a computer in algebra teaching
Michael Thomas, David Tall 601

67. The role of audiovisuals in mathematics teaching
Klára Tompa 609

68. Specifying the multiplier effect on children's solutions of simple multiplication word problems
L.Verschaffel, E.De Corte, V.Van Coillie 617

69. Is there any relation between division and multiplication?
Elementary teachers'ideas about division
Shlomo Vinner, Liora Linchevski 625

70. The influence of socialization and emotional factors on mathematics achievement and participation
Delene Visser 633

71. Metacognition and elementary school mathematics
Miriam A.Wolters 641

72. The development of the counting scheme of a five year old child: from figurative to operational
Bob Wright 649

73. Say it's perfect, then pray it's perfect: the early stages of learning about LOGO angle
Vicki Zack 657

74. Substitutions leading to reasoning
Nurit Zehavi 665

IV. LIST OF AUTHORS

PLENARY PAPERS

STREET MATHEMATICS AND SCHOOL MATHEMATICS

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There are different ways of summarizing one's own research. One is to retrace one's steps, to present a lived-through experience, with its excitement, its disappointments, its beliefs, its ideology, and its motives. It is a personal account about one's research. Another way is to choose a theoretical framework which not have anything to do with how you got to where you are in your work but which deepens the understanding of the questions you tried to address. My attempt here will be to do both--to set the stage for the analysis of street and school mathematics by following the first studies and their developments and then try to organize the findings by using a theoretical framework to sort out the similarities and differences between street and school mathematics. However, before I start, I must acknowledge that the research I will be reporting on resulted from a close collaboration with Analúcia Schliemann and David Carraher over seven years. I am certain that neither the excitement in doing this work nor the theoretical analysis which I am the presenter of here today could have come about without them.

I. THE STORY OF THE STUDIES

Brazil has a capitalist economy and a class structured society. Closely associated with this class structure is the phenomenon of school failure. Children from the dominant classes by and large are successful in school. In contrast, children from the working class fail in mathematics in school in high proportions. In Brazil as everywhere else, explanations for this class-related failure in schools evolved from blaming the victim--that is, asserting that the children were lacking in the required skills or in cognitive maturity and, for this reason, did not learn mathematics--to blaming the social system--that is, asserting that in a class-structured society schools are set up exactly to maintain the social structure, reproducing class differences in knowledge and culture.

These two very different approaches have one thing in common: they assume that children from the working class are less knowledgeable, less competent, for example, in mathematics, and that is why they fail. In either line of explanation, evaluations of children's mathematical competence are carried out from the school's viewpoint. Mathematical skills, it is believed, are neutral: two plus two is four both for the dominant and for the working class culture.

At first, we also thought so. We tried to find out why working class children were failing so often in mathematics by examining their basic competencies and the development of their learning of school mathematics (Carraher & Schliemann, 1983; 1985). We chose Piagetian concepts (such as conservation, class inclusion, and seriation) as ways of getting at what we thought were "basic", "universal" and "culture-free" competencies and chose some aspects of the local mathematics curriculum (ability to solve addition and subtraction problems) as ways of evaluating learning which had taken place in school. We wanted to evaluate their learning of addition and subtraction algorithms and see whether this learning correlated with cognitive development measures. However, when we set out to observe the children, they seemed to have their own ways of calculating (Carraher & Schliemann, 1985) and did not prefer the use of the algorithms we wanted to observe. When given freedom, they would rather use their own routes for solving problems; when using school-prescribed computation routines, they were likely to fail. Moreover, when children were allowed to solve problems in their own ways and when the type of curriculum offered by school was controlled for, differences between working class children and children from the dominant strata tended to disappear. Yet, in school more working class children failed, as usual--in our sample, 32% of the working class children failed arithmetic at the end of the year while only 2% of the middle class children failed.

These observations led us to question the school system's capacity to truly evaluate working class children's abilities in Mathematics although we still did not know why the system failed in this evaluation. Could it be that the development of mathenmatical skills is not a value-free question? Is Mathematics not an exact science, imune to the quibbles and quarrels of cultural relativism? Would not the evaluation of mathematical abilities be above the cultural question?

We wondered whether we could find ways to observe more clearly working class children's abilities in elementary mathematics. We thought of the fact that their families' income is often much too low for the family size and that children are then engaged in the informal economy to help their parents. They may, for example, sell fruits, vegetables, popcorn, candy, or refreshments. That means that they often have to calculate. If they truly lacked the elementary-school mathematical abilities, how could they handle their everyday life demands? From this informal observation of working class children's competence, we designed a study through which we compared five children's competence in everyday life with their competence in a school-like situation (Carraher, Carraher & Schliemann, 1982; 1985). Starting out as customers, we proposed purchases to the children and asked them about total costs of purchases and change if we gave them different notes as payment. Below is a sequence taken from this study which exemplifies the procedure:

Customer/examiner: How much is one coconut?
Child/vendor: Thirty-five.
Customer/examiner: I'd like three. How much is that?
Child/vendor: One hundred and five.
Customer/examiner: I think I'd like ten. How much is that?
Child/vendor: (Pause) Three will be 105; with three more, that will be 210. (Pause) I need four more. That is... (pause) 315...I think it is 350.
Customer/examiner: I'm going to give you a five hundred note. How much do I get back?
Child/vendor: One hundred and fifty.

When engaged in this type of interaction, children were quite accurate in their calculations: out of 63 problems presented in the streets, 98% were correctly solved. We then told the children we worked with mathematics teachers and wanted to see how they solved problems. Could we come back and ask them some questions? They agreed without hesitation. We saw the same children at most one week later and presented them with problems using the same numbers and operations but in a school-like manner. Two types of school-like exercises were presented: word problems and computation exercises. Children were correct 73% of the time in the word problems and 37% of the time in the computation exercises. The difference between everyday performance and performance on computation exercises was significant. These results convinced us of two things. First, street mathematics and school mathematics are not one and the same mathematics. Second, Brazilian schools do not acknowledge the existence of street mathematics, even if we all know of its existence through our everyday experiences. This appears to be an instance of what can be called the ideology of school mathematics: to ignore (or to treat as lesser mathematics) solutions which do not follow the school-prescribed ways.

The next study was provoked by other researchers' reactions to the findings. "This is not real mathematics, this is bricolage" was a comment we met up with quite often. A second comment was "the children think in concrete terms and thus do better in situations in which there are concrete materials". Our disbelief in these reactions was strong--but it was only a personal reaction. How could we find out whether street mathematics was the same as school mathematics in cognitive terms?

In our next study (Carraher, Carraher & Schliemann, 1987) 16 3rd-grade children were interviewed by a researcher in their own school. The researcher was introduced as a teacher interested in how children solve problems. Children were asked to solve problems in three different situations: a simulated store, word problems and computation exercises. By working in the school, we set up a situation in which the examiner-child relationship would be pretty much the same throughout testing. We arranged the groups of numbers and operation in a Latin

Square so that they were the same across situations for different children. By having a simulated store condition, we hoped that children would resort to their street strategies in solving problems. It seems that we were successful in reproducing the difference between street mathematics and school mathematics; problems in the simulated store and word problem conditions were correctly solved significantly more often than in the computation exercises condition, replicating the results of the previous study.

Children's **ways of solving problems** were influenced by the experimental conditions: the simulated store yielded between 80 and 89% of oral calculations (with percentages calculated by operation); the word problems yielded between 50 and 71%; and the computation exercises between 10 and 29%. There was also a strong difference in accuracy when problems were solved orally versus in writing. Table 1 shows the percentages of correct responses per operation solved correctly when children worked through oral or written calculation.

Table 1
Percentage of correct responses by operation and type of procedure used by the children (from Carraher, Carraher & Schliemann, 1987)

	Oral	Written
Addition	75	68
Subtraction	62	17
Multiplication	80	43
Division	50	4

When we controlled statistically for the type of strategy used by the children--that is, oral versus written--the differences between the situations tended to disappear.

This study convinced us that the effect of the situation upon children's performance was mediated by their choice of strategy. Social situations seem to determine how people solve problems and differences in strategies result in differences in performance. In school-like situations, particularly in computation exercises, children appear to believe that written mathematics is called for and try to use it even though they are less able with written than with oral mathematics. This interpretation is consistent with educational goals and educational practice in Brazil. In the simulated store, children used oral mathematics, which is a usual form of computation in street markets and serves both the purpose of personal representation and interpersonal communication. Vendors in street markets usually calculate for the customers the total price of their purchases and count the change up from the total to ensure that the customers recognize that they are receiving the correct amount of change.

The explanation for working class children's failure in mathematics in terms of lack of cognitive maturity no longer seemed plausible after these initial studies. The children were quite capable when calculating orally In contrast, the observation of their errors when they attempted computation in the written mode could be interpreted as resulting from "lack of comprehension". The protocol below is a clear example of comprehension in the oral mode although performance in the written mode seems suggestive of lack of ability:

> R., 8 years old, is solving 200 - 35 in the simulated store condition; he writes down 200 - 35 properly alligned in the vertical form and proceeds as follows:
> Child: Five to get to zero, nothing (writes down zero); three to get to zero, nothing; (writes down zero); two take away nothing, two (writes down two).
> Examiner: Is it right?
> Child: No! So you buy something from me, and it costs 35. You pay with a 200 *cruzeiros* note and I give it back to you?
> Examiner: Do it again, then.
> Child (after writing down 200 - 35 in the same form as above): Five, take away nothing, five (writes down five). Three, take away zero, three (writes down three). Two, take away nothing, two (writes down two). Wrong again!
> Examiner: Why is it wrong again?
> Child: Now you buy something, and it costs 35. You give me 200 and I give you 200 and 35 on top?
> Examiner: Do you know how much it is?
> Child: If it were 30, then I'd give you 170.
> Examiner: But it is 35. Are you giving me a discount?
> Child: One hundred and sixty five. (From Carraher, Carraher & Schliemann, 1987).

This study convinced us that lack of ability was not the correct explanation for the children's failure in school maths. Children's computation strategies in the oral mode contradicted their apparent lack of ability when they worked in the written mode. At first glance, however, oral procedures seemed idiosyncratic and disorganized--a feeling which has led authors in the past to disregard these methods. It looked as if each child designed each solution on the spot without any previous direction. However, when we looked at the strategies used, and not the specific steps carried out, there were two main ways of solving computations orally. Addition and subtraction were solved through **decomposition**; multiplication and division were solved through **repeated groupings**. These procedures are not exactly like school algorithms since the specific steps are not foreseen in the procedures. They are rather like the generative structures of language, which can generate an infinite number of sentences that are totally different in meaning but rest upon the same deep structures. In order to characterize this flexibility, we called oral procedures **heuristics** instead of algorithms. Table 2 presents an example of each of these heuristics.

Table 2
Examples of decomposition and repeated groupings
(From Carraher, Carraher & Schliemann, 1987)

Example of a solution through decomposition.
The child was solving the computation exercise "252 - 57".
Child: "Take away fifty-two, that's two hundred, and five to take away, that's one hundred and ninety-five".

Example of a solution through repeated groupings.
The child was solving a word problem which asked about the division of 75 marbles among five boys.
Child: "If you give ten marbles to each one, that's fifty. There are twenty-five left over. To distribute to five boys, twenty-five, that's hard. (Experimenter: That's a hard one.) That's five more each. That's fifteen".

However, we wondered, were we looking at atypical cases? Or is street mathematics a pervasive phenomenon? More and more we came to believe that the children we observed were not atypical. Pluncket (1979) had already mentioned the existence of decomposition. Reed & Lave (1981) also found similar mathematical procedures among the taylors in Liberia. Saxe & Posner (1983) documented procedures developed outside school in Papua New Guinea. Scribner and her co-workers (1984) found several instances of mathematical problem solving at work which did not follow the school-ways. Ginsburg (1982) found oral calculation in the Ivory Coast and in the United States. Further studies in Brazil added still more evidence: oral calculation through the same strategies was observed by Schliemann (1984) among carpenters, by Acioly and Schliemann (1986) among bookies taking bets, and by Carraher (1986) among foremen in construction sites.

A third study followed in pursuit of the origin of the knowledge of the numeration system so clearly displayed in decomposition and repeated groupings. We wondered whether dealing with money was a significant out-of-schoool experience which provided people with knowledge of the properties of the numeration system **in the absence of school experience**. Carraher (1985) worked with 72 pre-school children who had not yet received any instruction on writing numbers and 6 adults who had never attended school either in childhood or adult literacy programs. Their understanding of numeration systems was tested through questions about money either in verbal form only, in the case of the adults, or with the help of a play-money system, in the case of children. Three aspects of the understanding of number systems were tested: (1) the ability to differentiate between relative and absolute values (for example, comparing the total buying power of four coins worth one and four coins worth ten each); (2) the ability to decompose values within the decimal system (for example, paying 65 when you have nine coins worth ten and nine coins worth one, in the case of children, or figuring out what is the smallest number of notes needed to pay 365 using only bills of 100, 10 and 1, in the case of adults); and (3) the ability to write numbers using the place value system. The first two abilities, which relate to what we call number meanings, clearly preceeded the third, which refers to number writing. None of the children knew number notation although 44% of them were able to understand the distinction between absolute and relative value and 40% were able to combine different relative values to compose different sums of money. Among the adults, all were able to understand the distinction between absolute and relative value; four (out of six) were able to decompose numbers in hundreds, tens, and units; and only two appeared able to write numbers using place value. Through this study we were able to determine the existence of an oral comprehension of the basic meanings of numeration systems in the absence of knowledge of rules for writing numbers.

Now, what type of mathematics is oral mathematics? If people learn mathematical concepts in everyday life, are the resulting concepts different depending on whether they are learned outside of school without the assistance of an instructor or in school through explicit teaching?

To deal with these questions, we will need to refer to a theoretical framework which will be used in the comparison of concepts.

II. A COMPARISON OF STREET AND SCHOOL CONCEPTS: THE CASE OF ADDITION AND SUBTRACTION

Vergnaud (1985) has proposed a framework which we find very useful for the comparison of concepts learned in and out of school. According to Vergnaud, a concept necessarily entails a set of **invariants**, which constitute the properties defining a concept, a set of **symbols**, which are a particular way of representing the concept, and a set of **situations**, which give meaning to the concept.

Invariants in street and school mathematics

Taking addition and subtraction as an example, we can see that the invariants underlying street and school mathematics are the same; decomposition and written algorithms are based upon the property of associativity (see Resnick, 1986, and Carraher & Schliemann, 1988). In fact, if oral mathematics were to violate the properties of operations, results could not be correct. One example of decomposition will be reviewed below (Table 3) in order to illustrate that, despite great dissimilarities in the specific steps used in oral calculation, both oral and written addition/subtraction rest upon associativity.

Table 3
A comparison between decomposition and the subtraction algorithm

Computation: 252 - 57

Example of an oral procedure observed: "Just take the two hundred. Minus fifty, one hundred and fifty. Minus seven, one hundred and fourty three. Plus the fifty you left aside, the fifty two, one hundred ninety three, one hundred ninety five".

Steps used:
a: 252 - 57 =
b: (200 + 52) - (50 + 7) =
c: (200 - 50) + (52) - (7) =
d: (150 - 7) + (52) =
e: 143 + (50 + 2) =
f: 193 + 2 =
g: 195.

Written procedure prescribed in school: Two minus seven, you can't; borrow a ten. Five tens minus one ten, four. Two plus one ten is twelve; twelve minus seven, five. Four (tens) minus five (tens), you can't; borrow from the hundreds. Two hundreds minus one hundred, one hundred. Add one hundred to the four tens, fourteen tens. Fourteen (tens) minus five (tens), nine (tens). One (hundred) minus zero, one (hundered).

Steps used:
a: 252 - 57 =
b: (240 + 12) - (50 + 7) =
c: (240) - (50) + (12 - 7) =
d: (240 - 50) + 5 =
e: (100 + 140 - 50) + 5 =
f: 100 + 90 + 5 =
g: 195.

It can be easily recognized that both procedures work by a series of decompositions of the minuend and subtrahend and sequential operations according to the decomposition. The particular decompositions chosen in written and oral procedures are not the same; yet, the properties used, namely the associativity of addition and of subtraction, are the same in both procedures.

Symbolic representation and its impact upon street and school mathematics

Although street mathematics and school mathematics are based upon the same invariants, there are differences in the way subjects represent numbers across situations and solve problems--that is, there are differences in the symbols used in and out of school (see Carraher, Carraher & Schliemann 1987, Carraher & Schliemann, 1988, and Carraher & Carraher 1988). In oral mathematics, the relative meaning of the symbols is preserved while in written algorithms this relative meaning is set aside and operations are carried out upon the absolute value. In oral mathematics, calculation tends to run from hundreds to tens to units while in written mathematics calculation tends to go from units to tens to hundreds, with the exception of the division algorithm.

These differences can be understood (by analogy to language) as reflecting different processes for generating solutions to problems; while oral mathematics generates computation strategies on the basis of semantic relations, written mathematics generates solutions on the basis of rules for exchanging values from one column to another, as pointed out by Resnick (1982). These rules are in a sense similar to syntactic rules (like word order) but they are in some sense also different from syntax because they work in ways which are detrimental to meaning. The loss of meaning in written mathematics was clearly documented by Grando (1988) who asked 14 farmers who had little or no school instruction, 20 fifth graders and 40 seventh graders from the same region to solve some word problems and compared their problem solving strategies and intervals of responses, finding great differences between the groups. For example, they were asked how many pieces of wire with 1.5 meters of length could be obtained by cutting up a roll which had 7 m of wire in it. The farmers responses, obtained through oral calculation, fell between 4 and 7 pieces with 93% giving the correct answer. The students responses fell between 4 and 413 pieces. These extreme answers were given by students who carried out the algorithm for division (correctly or incorrectly) and did not know where to place the decimal point. The loss of meaning during calculation is likely to be responsable for the acceptance of answers which indicated that one could get more than seven pieces of wire out of the roll. These are absurd answers because anyone controlling for meaning would realize that 7 m divided into pieces of 1 m (ignoring the decimal point) yields 7 pieces; since 1.5 is bigger than 1, any response greater than 7 is not sensible. Among 5th graders 40% of the answers were greater than 7; 15% of the 7th graders produced such answers. In a second problem, subjects were told that one farmer had harvested 20% more soy this year than the year before and that his harvest in the previous year corresponded to 150 bags; they were asked to calculate how many extra bags the farmer harvested this year. Fifth-grade students had learned formulas for calculating percentages; 7th grade students had learned both formulas for calculating percentages and the proportions algorithm (a/b= x/c). Among farmers, 93% answered correctly using oral calculation and one gave an answer which we consider absurd since 30% cannot be more than 100% and his answer was 450. Fifth and seventh graders gave 30% and 35% correct responses, respectively. Their interval of responses varied between 3 and 3,000 bags; the extreme responses were again obtained through the use of algorithms coupled with failure to evaluate the meaning of the answer when one doesn't know where to locate the decimal point.

Grando's study also provides an analysis of how meaning is lost in more complex problems in the mathematization of the situation itself. She analyzed the implicit models used in mathematizing situations by looking at the sequences of operations performed and searching for their meaning in terms of the problem-situation. For example, subjects were asked to find out how many tea bushes were needed to fully plant a rectangular area 60 m x 40 m knowing that the space between the tea bushes has to be 4 m by 3 m. The problem was explained carefully to subjects with the help of a drawing in scale and the meaning of spacing the bushes in 4 by 3 explained to students in detail. She found that three models described the farmers' solutions and 13 were needed to describe the students' solutions. The models used by farmers were all meaningful, although two were incomplete in their analysis of the problem and, therefore, yielded wrong answers. One model consisted of finding out and then multiplying the number of rows by the number of columns of tea bushes; 58% of the farmers, 5% of 5th graders and 30 % of seventh graders used this model. A second involved obtaining the area of the rectangle and dividing it by the number of rows--an incomplete model, since a subsequent division by number of columns or a division by area needed by each tea bush would have been appropriate; 25% of the farmers used this model but no students did so. However, 5% of the 7th graders used the full model, dividing the total area by the area per tea bush. A third model, used by 8% of the farmers but not used by students, was the calculus of the number of rows or columns without completion of the solution. The remaining 95% of 5th graders and 65% of 7th graders attempted 11 different combinations of the four arithmetic operations, only one of which is amenable to interpretation (addition--instead of multiplication--of the number of rows and number of columns, a procedure consistent with these students' frequent conception of area as an additive relation of length and width).

These examples show that learning mathematics outside school does not always lead to correct responses even when the content of the problem is familiar. More importantly, they strongly indicate that even wrong responses tend to be obtained in sensible ways. In contrast, mathematics learned in school results in a high percentage of what we can call "basket" models in problem solving; many students just seem to throw the numbers and operations together in a basket instead of analyzing the meaning of the problem situations.

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The situations in which concepts are used in street and school mathematics

The differences in symbolic representation discussed above may result in yet other differences between street and school mathematics. Any representation stresses some aspects of what is being represented and leaves other aspects out of focus. If we consider representation as mediators of thinking, diverse forms of representation will result in diverging recognition of similarities and differences between problem situations. In oral mathematics, representation seems to be closer to the meaning of the problem situation; when working with written algorithms, we seem to move away from meaning in general and work more with relations between numbers. As a consequence, oral and written mathematics may apply their concepts to different sets of situations, thereby defining concepts of different extensions.

This possibility was explored with respect to the concepts of addition and subtraction by Carraher (1988) in a study with 90 adults attending night school, all normal and comopetent people in their own lives but who had no opportunity to attend school as a consequence of their socio-economic position. It was reasoned that their concepts of addition and subtraction would reflect their everyday experiences and not school learning if they were tested in their first year of school, when the curriculum emphasizes primarily reading instruction. As level of schooling increased, their concepts would probably approach the school concepts of addition and subtraction.

Previous studies (Carraher & Bryant, 1987; Carpenter & Moser, 1982) had suggested that children's strategies in solving addition and subtraction word problems change with grade level from attempting to represent the **situations** to representation of the **arithmetic operations** (that is, the numerical calculus). Everyday meanings for addition are the union of two sets (for example, "I have two yellow flowers and three red flowers") or an increase in amount (for example, "I had two flowers and got three from a friend"); the basic everyday meaning of subtraction seems to be a decrease in amount (for example, "I had three coins and lost one"). Problem solving strategies which represent the situations and those which represent the numerical calculus required for solution may result in the same or different conceptions of problem situations. For example, the problem "Mary had some stamps in her stamp collection. She got 27 stamps from her friend and now she has 32. How many did she have to begin with?" is conceived as a subtraction problem if we think of the calculation we would have to write down in order to carry out a computation algorithm leading to the answer; the computation would be 32 - 27. However, in oral mathematics one can simply count up from 27--28, 29, 30, 31, 32--keeping track of the number of fingers used while we counted up; we would find the solution through a process which represents what happened in the problem situation, an addition to the stamp collection. The school concept of subtraction used in this case is not that of a subtractive situation but one of subtraction as the inverse of addition. For this reason, I will refer to problems such as this as "inverse problems". In contrast, in "direct problems" the numerical calculus required for solving the problem and the representation of the situation call into play the same operation.

Comparison problems, which are also related to addition and subtraction, seem to be solved by children before they have had much schooling through matching strategies (see Carpenter & Moser, 1983). As long as this type of strategy is the only one available to the children, it appears to remain separated from the concepts of addition and subtraction; subjects can solve the problems but do not know which operation to use since their solution is obtained neither by addition nor by subtraction. Would unschooled adults recognize the operations called for in solving comparison problems or would their strategies reflect more a representation of the situation than a representation of the operation?

The adults interviewed in this study were in three grade levels, having had one, three or five years of school instruction. They were asked to solve three direct addition/subtraction problems, three inverse problems and two comparison problems. They were given a calculator in order to avoid calculation errors, were taught how to carry out the four arithmetic operations with the calculator and were encouraged to use it during the experiment. Numbers in the problems were two-digit numbers to avoid memorized solutions and make choice of operation instrumental to problem solving. Two dependent variables were examined as a function of problem-type: accuracy of solution and choice of operation. Tables 4 and 5 summarize the results of this study. The results for comparison problems which pose the question "how many fewer?" and for those with the question "how many more?" were separated because "fewer" and "minus" are expressed by a single word in Portuguese (menos) and "more" and "plus" are also expressed by a single word (mais).

Table 4
Percentage of correct responses
by problem type and grade level

Problem type	Grade level		
	First	Third	Fifth
Direct	78	92	92
Inverse	40	58	67
Comparison (how many fewer?)	56	87	77
Comparison (how many more?)	40	67	67

We can see that each group of subjects did consistently better when the everyday concept coincided with the numerical calculus than when it did not--an observation which indicates that the school concept of addition/ subtraction is not fully accomplished by all adults even at the fifth grade level.

Table 5 indicates the percentages of choices of (a) the correct operation according to the school model of the problem, (b) the inverse operation, and (c) the use of another strategy (e.g., counting) as a function of problem type. It is easy to recognize that the same trends observed with respect to correct responses are obtained when choice of operation is taken as the dependent variable: third- and fifth grade adults choose the operation consistent with the school model more often than first-grade adults.

Table 5
Percentage of choice of operation
by problem type and grade level

Problem type	Grade level		
	First	Third	Fifth
Direct	Correct 89	Correct 93	Correct 94
	Inverse 8	Inverse 3	Inverse 2
	Other 3	Other 4	Other 4
Inverse	Correct 47	Correct 57	Correct 57
	Inverse 50	Inverse 35	Inverse 35
	Other 3	Other 8	Other 8
Comparison (how many fewer?)	Correct 74	Correct 86	Correct 86
	Inverse 22	Inverse 11	Inverse 7
	Other 4	Other 3	Other 7
Comparison (how many more?)	Correct 38	Correct 57	Correct 60
	Inverse 52	Inverse 33	Inverse 33
	Other 10	Other 10	Other 7

Despite the fact that adults with very little schooling can correctly solve additions and subtractions, displaying an understanding of the same invariants that define school concepts, they use concepts of different extensions, because their concepts are applied to a set of situations which differs from the concepts we teach in school. Addition and subtraction concepts as we use them presently in school are a specific, culturally developed way of conceiving problems which has to be learned by people even if they already understand the basic invariants of addition and subtraction.

III. CONCLUSIONS

Summing up this discussion of research about street and school mathematics, some similarities and differences will be pointed out.

Many invariants of mathematical concepts taught in school appear to be quite basic and necessary for solving problems in everyday life. These invariants can be understood outside school, without the benefit of teaching, through the understanding of problem-situations. These basic invariants are perhaps analogous to core semantic structures of language; their understanding can generate solutions to problems as semantic structures can generate linguistic expression. However, linguistic expression is not only a matter of meaning but is a matter or grammar also--particular grammars of particular languages. In mathematics, like in language, we must deal with particular and arbitrary ways of representing mathematical meanings. When linguistic expression violates the specific grammatical rules of our language, as it happens when a foreigner speaks to us, we have difficulty in finding the meaning. Similarly, when mathematical solutions to problems deviate from the conventional ways, they are hardly ever recognized as appropriate by teachers.

Granting the similarities in invariants of concepts learned in and out of school, let us sum up the differences.

(1) Brazilian street mathematics is oral both in the sense that it is spoken and in the sense that it is used for communication. School mathematics is written; it is not chosen for communication but for the transmission of culturally developed ways of thinking and representing concepts and, perhaps more importantly for the functioning of schools, for the evaluation of individual children's work. Correct responses given in tests without the "proper" written calculation receive at best partial recognition in Brazil.

(2) Mathematics learned outside school is a tool for solving problems in meaningful situations. In school we teach mathematics as an object; applications, when used in the classroom, tend to come after the teaching of the model.

(3) Mathematics learned outside school is conducive to the development of problem solving strategies which reveal a representation of the problem situation. The choice of models used in problem solving and the interval of responses are usually sensible even though not always correct. Students using their school mathematics often do not seem to keep in mind the meaning of the problem, displaying problem solving

strategies which have little connection with the problem situation and coming up with and accepting results which would be rejected as absurd by anyone concentrating on meaning.

The meanings of problem situations are not always absent in representations in school mathematics. The difference between fractions and percentages, for example, is mostly a difference between general part-whole relationships, which are treated as fractions, and a specific situation, in which the whole is equal to 100. However, the similarity of the invariants underlying the two concepts is not usually pointed out to pupils in Brazil. School mathematics separates fractions and percentages as different topics with different written procedures for finding equivalences; although meaning is in some sense a part of the separation of fractions and percentages as different topics in mathematics, the emphasis during teaching is still placed upon rules.

(5) Representation of meaning of problem situations may result in different extensions for concepts developed in and out of school. When cultural artifacts--such as calculators--embody the school concept, they may be of little use to those who have learned mathematics only outside school.

The differences pointed out above have some implications for mathematics education. Building bridges between street and school mathematics appears to be a route worth investigating in education. These bridges may sometimes be built through finding out what pupils already know from their out-of-school mathematics curriculum and let them use and expand this knowledge in school. Sometimes they may be built by using in school problem situations which can be analized and understood by pupils without focusing so much on rules. These are routes which have not been explored enough so far.

However, there is one issue in mathematics education which can be sorted out on the basis of these studies. I think that it is now clear that mathematics can no longer be treated as the gate-keeper which sorts out the academically able from those who are not gifted enough. Any normal child must be treated as capable of learning elementary school mathematics--especially if we are able to discover situations in which the mathematical properties we want children to understand are genuine parts of situations that we present to children and allow them to master over time while we make available to them representations current in our culture of mathematics classrooms. If we succeed in doing this, our pupils may become skilled in analyzing problem situations, in generating meaning-sensitive plans to solve problems, in appropriating for their own use our mathematical representations and ideas, and may actually come to enjoy mathematics and see it as valuable from their own perspective.

References

Acioly, N. & Schliemann, A. D. Intuitive mathematics and schooling in understanding a lottery game. **Proceedings of the Tenth PME Conference**, London, July, 1986.

Carpenter, T. P. & Moser, J. M. The development of addition and subtraction problem-solving skills. In T. Carpenter, J. Moser & T. Romberg (Eds.), **Addition and Subtraction: A Cognitive Perspective**. Hillsdale, NJ: Lawrence Erlbaum, 1982, 9-24.

Carraher, T. N. The decimal system. Understanding and notation. In L. Streefland (Ed.), **Proceedings of the Ninth International Conference for the Psychology of Mathematics Education**, Vol. 1. Utrecht, The Netherlands: University of Utrecht, Research Group on Mathematics Education and Educational Computer Centre, 288-303.

Carraher, T. N. From drawings to buildings: Working with mathematical scales. **International Journal of Behavioral Development**, 1986, 9, 527-544.

Carraher, T. N. Adult mathematical skills. The contribution of schooling. Paper presented at the annual meeting of the American Education Research Association, New Orleans, April, 1988.

Carraher, T. N. & Bryant, P. E. Everyday and school concepts of addition and subtraction. Paper presented at the China Satellite Conference of the International Society for the Study of Behavior Development, Peking, July, 1987.

Carraher, T. N. & Carraher, D. W. Mathematics as personal and social activity. **European Journal of Education**, 1988, in press.

Carraher, T. N., Carraher, D. W. & Schliemann, A. D. Na vida, dez. Na escola, zero. Os contextos culturais da aprendizagem da matemática. **Cadernos de Pesquisa**, 1982, 42, 79-86.

Carraher, T. N., Carraher, D. W. & Schliemann, A. D. Mathematics in the street and in schools. **British Journal of Developmental Psychology**, 1985, 3, 21-29.

Carraher, T. N., Carraher, D. W. & Schliemann, A. D. Written and oral mathematics. **Journal for Research in Mathematics Education**, 1987, 18, 83-97.

Carraher, T. N. & Schliemann, A. D. Fracasso escolar. Uma questão social. **Cadernos de Pesquisa**, 1983, 45, 3-19.

Carraher, T. N. & Schliemann, A. D. Computation routines prescribed by schools: Help or hindrance? **Journal for Research in Mathematics Education**, 1985, 16, 37-44.

Carraher, T. N. & Schliemann, A. D. Culture, arithmetic, and mathematical models. **Cultural Dynamics**, 1988, in press.

Ginsburg, H. The development of addition in contexts of culture, social class, and race. In T. P. Carpenter, J. M. Moser & T. A. Romberg (Eds.), **Addition and Subtraction: A Cognitive Perspective**. Hillsdale, NJ: Erlbaum, 1982, 99-116.

Grando, N. A matemática no meio rural: pequenos agricultores, estudantes e professores. Masters' Thesis, Mestrado em Psicologia, Universidade Federal de Pernambuco, 1988.

Hatano, G. Adult mathematical skills. Symposium presented at the annual meeting of the American Educational Research Association, New Orleans, April, 1988.

Plunkett, S. Decomposition and all that rot. **Mathematics in schools**, 1979, 8, 2-7.

Reed, H. J. & Lave, J. Arithmetic as a tool for investigating relationships between culture and cognition. In R. W. Casson (Ed.), **Language, Culture, and Cognition: Anthropological Perspectives**. New York: McMillan, 1981, 437-455.

Resnick, L. B. Syntax and semantics in learning to subtract. In T. P. Carpenter, J. M. Moser & T. A. Romberg (Eds.), **Addition and Subtraction: A Cognitive Perspective**. Hillsdale, NJ: Erlbaum, 1982, 136-155.

Resnick, L. B. The development of mathematical intuition. In M. Perlmutter (Ed.), **Minnesota Symposium on Child Development**. Vol. 19. Hillsdale, NJ: Erlbaum, 1986.

Saxe, G. & Posner, J. The development of numerical cognition: Cross-cultural perspectives. In H. Ginsburg (Ed.), **The Development of Mathematical Thinking**. New York: Academic Press, 1983.

Schliemann, A. D. Mathematics among carpenters and carpentry apprentices: Implication for school teaching. In P. Damerow; M. E. Dunkley; B. F. Nebres & B. Werry (Eds.), **Mathematics for All**, UNESCO, Science and Technology Education Document Series, number 2562, 1984, 92-95.

Scribner, S. Studying working intelligence. In B. Rogoff & J. Lave (Eds.), **Everyday Cognition: Its Development in Social Context**. Cambridge, MA: Harvard University Press, 1984, 9-40.

Vergnaud, G. Concepts et schemes dans une théorie opératoire de la répresentation. **Psychologie Française**, 1985, 30 (3/4).

A LOOK AT THE AFFECTIVE SIDE
OF MATHEMATICS LEARNING
IN HUNGARIAN SECONDARY SCHOOLS

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The aim of this presentation is to provide an empirical survey of some facts of Hungarian mathematics education and its psychological conditions.

MATHEMATICS EDUCATION IN HUNGARY

Hungary has for centuries had a good reputation of being a land which breeds great innovators of mathematical thought. It may be well known also that several outstanding personalities of mathematical learning and problem solving, as well as mathematics teaching, started their career as Hungarians - from György Pólya to Zoltán P.Dienes, to mention just a few.

The everyday setting and process of mathematical education at primary and secondary levels, however, does not always confirm expectations based on that fame. There is great diversity allowing for good achievement and severely disadvantaged groups and practices.

The historical determinants of the present-day situation can be traced back to 1962 when an international conference in mathematics learning took place in Budapest. Among other contributors, Z.P.Dienes explained to participants the mathematical, psychological, and educational principles of a renewed mathematics teaching. Encouraged by the scientific activity of Dienes and others, Tamas Varga undertook the formidable work of introducing one of the best balanced methods inside the "New Mathematics" movement experimentally. Later, it was subjected to psychological measurement. The method came to be known as the 'composite method' or the 'O.P.I.[3] Mathematics' project. It is well documented in educational literature (VARGA, 1962; 1964; 1965; 1967; 1972).

Small scale experiments being carried out from the early 1960s to 1974 led to a nationwide implementation, in eight-grade Hungarian elementary school, of the Varga curriculum and metnod. It replaced the conventional curriculum in a gradual fashion, reaching the final grade of elementary school in 1980 (VARGA, 1987).

The 'O.P.I. method' succeeded in combining radical, even avantgardistic, components with ones satisfying traditional requirements and needs of teaching arithmetic (cf. KLEIN, 1987). Although it is a rare phenomenon that such an innovative system of ideas breaks through and becomes implemented nationally, that meant a transformation of the project as well. Practicing teachers and especially teacher training institutions were unable to fully substantiate new ideas in their practical programmes, thereby causing an at least initial loss of several important ingredients. Some authors even describe a deterioration of original advantages after the method was embodied in a central curriculum. The original aims of the 'composite' method stressed -- apart from content-related objectives -- psychological priorities like enhancement of creativity, increasing student motivation, building favourable attitudes toward mathematics, reducing school anxiety, developing linguistic--communicative skills, and the like. Including novel subject matter (probability, statistics, mathematical logic, vector spaces, topology etc.) in curricula was seen by innovators as a basis of presenting students thought-provoking, challenging tasks. The objectives of the 'O.P.I. method' were not, however, meant to be solely realised by content change; rather, by enriching classroom methods of teaching (e.g. discovery method, cooperation in small student groups, experiential learning of concepts from concrete embodiment, etc.)

During nationwide implementation, content areas typical to "New Mathematics" were being reduced. Deterioration of the whole process was mainly characterized, however, by inadequate application of classroom teaching methods. We were aware that changing of methods necessitated deep changes in teacher personality, a rearrangement of several value and attitude structures, and dissolution of habituated behaviours. The teacher the new process envisaged was, after all, a professional in a different sense: facilitator and animator of student learning, instead of a transmitter of mathematical knowledge and ready-made solutions.

Our primary interests as psychologists in a number of consecutive projects (1969--1987) were

(i) to describe psychological effects of Varga's method and of the nationally implemented version (involving effects on abilities, skills, attitudes, orientations, motivational factors, and holistic patterns of personality);

(ii) to verify the psychological hypotheses derived from the innovative programme concerning distinctive effects on children exposed versus not exposed to the O.P.I. method (or its later implementation).

Extensively reported findings on elementary school mathematics learning could be summarized by saying that only classrooms where pedagogical principles of the 'composite' method were genuinely followed could give predicted outcomes in personality and performance. Statistically significant psychological changes could be expected to occur where educationally significant changes in teaching and learning were observed.

SOME EMPIRICAL FINDINGS

In the following the scope will be limited to some empirical data concerning final grades of secondary schools. They constitute only a selection of simple descriptive results. As opposed to primary education, secondary schooling is not compulsory in Hungary, which may imply a practical possibility of higher niveau and greater divergence (among schools, programmes, classrooms, or among individuals). Secondary mathematics education was able to follow the wake of modernizing in primary mathematics with a substantial delay. In recent years, freedom of secondary institutions to decide on curricular options, enrichment, special programmes etc. was increased markedly. That may be conducive to faster changes in methods and results of secondary mathamatics education as well.

The two main types of secondary school in this country are grammar schools and 'specialized vocational secondary schools' (hereinafter 'SVSS'). Other types rarely offer four-year courses and almost never a certification of maturity entitling the student to apply to enter a university or higher college.

Data in the following sections are taken from the system RMPP, extension of the Hungarian (Population 'B') data set of the Second International Mathematics Study of IEA[4/]. All results are based on nationally representative samples.

A portrait of teaching: Time budget of activity forms

A relatively detailed and reliable picture of classroom activities can be drawn from time budget analyses. Table 1 shows allocations of time (measured in minutes per week) for specific pedagogical activities. These data stem from teacher estimates. The teacher sample is not only representative but quite heterogeneous along several dimensions. Here follow some descriptive figures also to characterize the population of secondary mathematics teachers in Hungary. The age of teachers in the sample varies from 26 to 64 years (average 40.2 years). Their practice in school ranges from 1 to 43 years (average 16.2), while practice in fourth grade, the measured student cohort, from 1 to 35 years (average 10.0). They had received 8 to 27 terms (semesters) of higher mathematical and pedagogical education; the average is 9.2 semesters.

Hungarian secondary school mathematics teachers are overstressed. They have to teach an average of 25.4 periods per week (45 minutes each), of which 18.5 are periods of mathematics. The other subjects are usually physics or other natural science subjects. A teacher is normally requested to lecture in two fourth classes. The common practice, however, is that most teachers teach lower secondary school classes as well, the average number of which is 3.1. Consequently, an average mathematics teacher has to be in contact with 5 different classes; as the usual number of students is over 30, this means a total of 150 children. Such an amount of work is done in an environment where heavy duties (mainly clerical) outside classroom jobs are burdened on teachers, and salaries are significantly lower than that of skilled factory workers.

Table 1 gives the original amounts of time estimated by teachers for ten predefined categories of activity ('Preparation for classroom work outside class', 'Grading student papers and tests', etc.). Each estimate was obtained once for the last full week before data collection and once for a "typical", average week. The estimates may be biased by the teacher's general tendency of over- or underestimation.

In Table 2, proportions (%) of the same durations are given as compared to sum totals of three larger blocks of responses (separated by horizontal lines). Each estimate was obtained once for the last full week before data collection and once for a

TABLE 1

Number of minutes allocated by teachers to specific catagories of activity

Hungarian secondary school teachers, N = 94

	$\bar{x}$	
	Week before data collection	"Typical", average week
Preparation and planning outside classroom	134.8	144.9
Grading student papers, quizzes, tests outside class	142.3	98,2
Σ	277,1	243.1
Explaining mathematics content new to the class	18.0	74.5
Reviewing mathematics content not new to the class	127.3	77.0
Routine administration	14.7	16.0
Establishing class order, disciplining students	3.7	4.9
Σ	163.7	171.4
Giving tests and quizzes to whole class	34.2	29.6
Individual student work (seat or blackboard work) in class	70,3	67.2
Frontal lecturing and explanation	44,1	64.1
Small group work in class	12.6	15,6
Σ	167.2	176.5

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TABLE 2

Proportion of time allocated by teachers to specific categories of activity

Hungarian secondary school teachers, N =94

	$\bar{x}$ (of % Proportion)	
	Week before data collection	"Typical" average week
Preparation and planning outside classroom	48.6	59.6
Grading student papers, quizzes, tests outside class	51.4	40.4
Σ	100.0	100.0
Explaining mathematics content new to the class	11.0	43.5
Reviewing mathematics content not new to the class	77.8	44.9
Routine administration	9.0	9.3
Establishing class order, disciplining students	2.3	2.9
Σ	100.0	100.0
Giving tests and quizzes to whole class	20.5	26.8
Individual student work (seat or blackboard work) in class	42.0	38.1
Frontal lecturing and explanation	26.4	36.3
Small group work in class	7.5	8.8
Σ	100.0	100.0

"typical", "average" week. The estimates may be thought to be biased by teachers' general tendency to over- or underestimate. Therefore, percentages are more reliable than absolute values.

Time assigments concerning last week are more or less correct reports of what happened the week before. "Typical week" ratings may reflect normative elements more than factual memory. As shown by data from "last week" assignments, e.g., teachers spent more time with grading student work than with preparing for classroom periods. In "typical" weeks, however, they reported to spend much more time in preparation than in reviewing student work.

During preceding week, teachers spent about seven times more minutes with reviewing material already taught; in the pattern of "typical" week, newly explained and reviewed material would consume approximately the same amount amount of time (Table 2).

Outside-class activities increase the already mentioned large number of periods (~25 hours per week) by at least 4 additional hours. Half of the latter are spent by preparation, another half by grading papers and tests. It is quite clear that it would be advisable to allocate more time for preparation than for grading.

Frontal explanations are certainly not the best way of teaching mathematics. Further, if we accept data on "previous week" as reliable it is striking that the proportion of reviewing to exposing new material is so high.

Although teachers estimated administration and disciplining at low levels in their time budgets, classroom and school observations as well as case studies seem to indicate that the last two forms of activity occur more frequently than preferable. These are forms of activity constantly discussed: they have been, on one side, managerially required, on the other, stigmatized, in the last decade. Teachers may have been over-anxious in giving their proportions. The low figure (~ 8%) obtained for small group work appears, however, like an overestimation when relying on classroom observation data.[5/]

Mathematical skills and abilities in students

As the main focus of the empirical part of this paper will be on attitudes toward mathematics in students, and their relation to teacher attitudes, only a brief point is made concerning student skills. If standardized, rotated mathematics tests of IEA (carefully adapted to Hungarian terminology, curricular content, and teaching traditions) are accepted to measure 'mathematical skills' or 'ability', results show that the student population is divided between extremes. Table 3 shows averages of TTGTD, an overall score, for various groups of institution and curriculum types. TTGTD is an average of two full-test indices computed over weighted item scores. The average skill level of the enriched Mathematics 11 grammar school curriculum group is about five times higher than the Kindergarten nursing SVSS (D) mathematics curriculum group.

Interestingly, these enormous differences do not stem from total unfamiliarity of sections of mathematical content. Items of rotated tests were evaluated from the viewpoint of whether the student was exposed to the specific content of the item or not, and if yes, in the academic year of measurement or before. Items judged 'certainly taught' were more numerous in SVSS curricular groups 'C' and 'D' than in any other curricular group for the year of measurement (RADNAI-SZENDREI and HABERMANN, 1984). As for the previous years, 'certainly taught' items were only slightly more numerous in Mathematics 'I' and 'II' of grammar school than in other curricular categories.

Attitudes of students toward mathematics

A point to be made in somewhat more detail in this paper is the role of affective (student and teacher) characteristics in Hungarian mathematics education, a problem in the centre of empirical investigations within 'Second International Mathematics

TABLE 3

Difference of curricular sets of cases on an overall performance measure of rotated standardized mathematics tests

Hungarian secondary school students, $N \sim 2450$

	$\bar{x}$
Grammar schools Curriculum "Mathematics II" (special)	13.41
Grammar schools Curriculum "Mathematics I" (special)	8.68
Specialized vocational secondary schools (SVSS), Curriculum F (Direction Computing services)	6.03
Grammar schools Basic curriculum	5.41
SVSS, Curricular group A/B/E (Direction Industrial and agricultural professions)	4.87
SVSS, Curriculum C (Direction Health and medical professions)	3.39
SVSS, Curriculum D (Direction Kindergarten nursing)	2.66

By two-tailed t-test, any pair of the above curricular groups shows a difference along achievement scores at $p < 0.001$ except Grammar schools/Basic versus SVSS/Curriculum F where the difference is significant at the $0.05 > p > 0.01$ level.

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Study' and 'RMPP' data systems. Mathematics has for a long time been a terrain to discover patterns of affective relations toward a school subject and its related activities (DUTTON, 1954, DUTTON and BLOM, 1968; MAERTENS, 1968, etc.). Separated from variables characterizing overall motivational states and processes of students, school subject specific affective variables were investigated in terms of 'interests', 'beliefs', 'views', 'opinions', 'preferences' and 'attitudes'.[6/] As BLOOM (1976) has noted there is no clear conceptual demarcation among any pair of these constructs. All of them presuppose, however, 'a continuum ranging from positive views, likes, or positive affect toward a subject to negative views, dislikes, or negative affect toward the subject' (BLOOM, op.cit., p.77). One can devise more or less parsimonious sets of concepts and operationalizations for such a continuum. In his 'Model of School Learning' (CARROLL, 1963) and even its extensions (e.g., 1984) John B.Carroll united affective characteristics under the name of 'perseverance', operationalized by duration of time the learner is willing to spend in learning[7/]. Following empirical investigations, BLOOM (1976) could build a conceptually richer model in which affective entry characteristics are subdivided into constructs termed 'subject-related affect', 'school-related affect' and 'academic self-concept'[8/]. When using 'attitude' (toward mathematics) below, we would not like to suggest anything specific in social-psychological definitions of that term. Rather, we shall apply the neutral term 'opinions about mathematics' for item variables (attitude scales) and the term 'attitude' simply for empirically verified higher-order structures from these measured items. In this we follow the general terminology of IEA studies (KIFER,1979).

In the following section, univariate descriptive data of two different sets of attitudinal (opinion)[9/] variables concerning mathematics will be presented. First, affective components related to forms of mathematical activity in school will be treated. Second, components pertaining to mathematics as a subject and as a science at a more general level will be discussed.

The affective instrument of IEA Second International Mathematics Study identified 15 forms of mathematical activity most commonly encountered in school life. These were: 'Checking an answer to a problem by going back over it'; 'Memorizing rules and formulae'; 'Solving word problems'; 'Getting information from statistical tables'; 'Solving equations'; 'Proving theorems'; 'Using vectors'; 'Working with complex numbers'; 'Investigating sequences and series'; 'Differentiating functions'; 'Drawing graphs of functions'; 'Finding a limit of a function'; 'Integrating functions'; 'Determining the probability of an outcome'; 'Using a hand-held calculator'. Opinions concerning each of the above activity forms were judged by students along three dimensions: Importance (important-unimportant), Difficulty (easy--difficult), and Preference or Likedness (liked--not liked).

Table 4 selects among the Importance ratings only those categories of activity which were assigned "extreme" values. Hungarian secondary school students judge the memorization of rules, getting information from statistical tables, and checking answers as most important mathematical doings within school. Only one item exceeded the conceptually negative limit 3.5 (near to the scale pole 'not important at all'), i.e. the use of hand-held calculators. At the time of measurement, portable calculators were already relatively available and inexpensive in Hungary, but in-school calculator use was at many places discouraged or forbidden.

If the Difficulty ratings are ordered (Table 5) calculator use occupies the first place among easiest activities. The next two forms, 'Checking answers' and 'Drawing function graphs' are, however, among the ones rated very important earlier. Two items similarly high-placed in the importance group, 'Proving theorems' and 'Solving word problems' are listed as judged most difficult. There is no stereotyped inference, therefore, in student thinking that it is the set of subjectively difficult activities which are really important. Whereas 9 out of 15 activities were evaluated rather important, only 3 of the same exceeded the limit among difficulty ratings (easiest items). The great majority of activity forms, at least when analysed over the entire sample,

TABLE 4

Attitudinal statements : Ratings of importance
Hungarian secondary school students, $N \sim 2450$

1 , most important, ... 5 , least important
15 activity forms rated

FORMS OF MATHEMATICAL ACTIVITY Rated most important ($\bar{x} < 2.5$)	$\bar{x}$
Memorizing rules and formulae	1.69
Getting information from statistical tables	1.84
Checking an answer to a problem	1.89
Proving theorems	2.20
Drawing graphs of functions	2.26
Solving word problems	2.26
Integrating functions	2.38
Differentiating functions	2.39
Finding a limit of a function	2.48
Rated least important ($\bar{x} > 3.5$)	
Using a hand-held calculator	3.55

TABLE 5

Attitudinal statements : Ratings of difficulty
Hungarian secondary school students, $N \sim 2450$

1 , easiest, ... 5 , most difficult
15 activity forms rated

FORMS OF MATHEMATICAL ACTIVITY Rated easiest ($\bar{x} < 2.5$)	$\bar{x}$
Using a hand-held calculator	1.68
Checking an answer to a problem	2.29
Drawing graphs of functions	2.45
Rated most difficult ($\bar{x} > 3.5$)	
Proving theorems	3.83
Solving word problems	3.55

are perceived neither peculiarly difficult nor easy by students.

Still less of the activity forms reached pronounced likedness or dispreference (Table 6). Calculator use (an item already identified as least important and easiest among all) is the most preferred. Solving equations is another form markedly preferred. Proving theorems and memorizing rules/formulae are the items looked upon with aversion (cf. next section on more general opinions).

Apart from mathematical activity forms, statements describing personalized positions concerning mathematics at a more general[10] level were also judged by students, in the format of modified Likert scales. Statements unequivocally agreed and disagreed with are listed in Table 7. The five statements which showed widest consensus can be divided among three attitude areas. Two of the statements relate to mathematical self-concept (cf. BROOKOVER; SHAILER and PATTERSON, 1964; FARQUHAR and CHRISTENSEN, 1967; HELMKE, 1987, 1988) expressing aspiration to good achievement and placing self-directed learning at a high value. One dimension relates to gender problems, while the remaining two, to beliefs concerning mathematical knowledge. Students hold that mathematics 'helps to think logically' and that there is 'more than one solution' to most problems. They disagree with several statements in direct connection with the ones agreed upon. Two statements negated in rating are opposites of the gender statement in the agreed group. The assertion that one would not voluntarily learn mathematics is expressing an idea opposed to the agreed-upon statement 'I really want to do well in mathematics'. At a basic level, these pairs of judgements corroborate each other and point at the reliability of not only the instrument as such but of interpreting item-level data as well. Students mostly are not afraid of mathematics. They have reservations, however, as to the everyday utility of mathematics and practice dispensing with the use of rules (trial and error). They construe mathematics as more than something 'to be memorized'. The last two items are not necessarily in contradiction. As the previous group of items revealed, Hungarian secondary school students judge 'memorizing rules' as highly important. At the same time, they disprefer

TABLE 6

Attitudinal statements: Ratings of preference
Hungarian secondary school students, N $\sim$ 2450

1, most liked, ... 5, least liked
15 activity forms rated

FORMS OF MATHEMATICAL ACTIVITY

Rated most liked ($\bar{x} < 2.5$)	$\bar{x}$
Using a hand-held calculator	2.30
Solving equations	2.31

Rated least liked ($\bar{x} > 3.5$)	
Proving theorems	3.65
Memorizing rules and formulae	3.55

TABLE 7

Attitudinal statements about mathematics
(as a subject and as a science)
Hungarian secondary school students, N $\sim$ 2450

1, strongly disagreed with, ...
5, strongly agreed with
45 statements rated

STATEMENTS

Agreement strongest ($\bar{x} > 3.75$)	$\bar{x}$
I feel good when I solve a mathematics problem by myself	4.43
A woman nees a career just as a man does	4.43
Mathematics helps one to think logically	4.33
There are many different ways to solve most mathematics problems	3.89
I really want to do well in mathematics	3.81

Disagreement strongest ($\bar{x} < 2.25$)	
Learning mathematics involves mostly memorizing	2.04
It scares me to have to take mathematics	2.09
Boys need to know more mathematics than girls	2.10
Mathematics is needed in everyday living	2.11
If I had my choice I would not learn any more mathematics	2.19
Men make better scientists and engineers than women	2.19
In mathematics, problems can be solved without using rules	2.22

memorizing and hold that learning mathematics should extend that, probably in a direction of more creative thinking. Disagreement with the statement expressing unnecessity of rule application in problem solving is consistent with the (at least abstract) importance students assign to memorizing (and thereby to using) rules.

Differences between types of institution along opinion scales

Recalling that mathematical achievement on standardized tests differentiated grammar school and SVSS pupils, and confirmed extreme ability gaps, these two large subpopulations were compared along affective components as well. As predicted, there were significant differences between the two subsamples along almost every dimension. Out of 46 primary variables of the "general" domain of attitude toward mathematics, for instance, only 10 opinions did not distinguish significantly between the two sets of cases when examined by a two-tailed *t* test. Among these, 27 variables showed a difference significant at $p < 0.001$ level. Vocational school pupils invariably have poorer (less acceptable) views on mathematics. They agreed less, or disagreed more, than grammar school students with statements like 'There are many different ways to solve most mathematics problems', 'Mathematics helps one to think logically', 'Mathematics is useful in solving everyday problems', and especially with statements of the mathematical self-concept. 'I really want to do well in mathematics', 'I feel good when I solve a mathematics problem by myself', 'I usually understand what we are talking about in mathematics', 'I like to help others with mathematics problems', 'I feel challenged when I am given a difficult mathematics problem', 'Working with numbers makes me happy', 'I usually feel calm when doing mathematics problems', and several others are statements to which -- whatever the mean value of agreement is -- SVSS students can consent significantly less ($p < 0.001$) than grammar school students.

The relation of attitudes toward mathematics to mathematical skills

Another way of characterizing processes in Hungarian mathematics education is to analyse the extent to which the attitudinal components just discussed may contribute to levels of mathematical 'skills' and 'abilities'. In this paper, only some elementary data concerning this relation will be presented. Mathematical skills and abilities will be operationalized, like before, by the overall score TTGTD of two rotated standardized IEA mathematics tests.

The relations of attitude components, pertinent to activity forms, to skill levels are demonstrated in Table 8. It is no surprise that closer correlations are all negative as attitude scales are conceptually inverted (small values represent important, easy and preferred activity items). At the very high number of cases in the study all Pearson correlation coefficients over $|0.10|$ are significant at the $p < 0.001$ level. The selective table lists only those under -0.25. Even then, correlation coefficients allow to state that only a relatively small proportion of ability (skill) variance can be explained by the attitude items. Nevertheless associations make educational sense. Solving word problems and proving theorems are two of the ubiquitous activities of school mathematics. Students who prefer these forms of activity (low attitude value) tend to have higher ability, and the reverse, students with better abilities can find it natural to prefer these mathematical tasks. The latter applies to the Difficulty correlation as well (higher ability students find word problems easier).
The more general domain of attitudes (Table 9) produced higher ability correlations. Persons with favourable mathematical self-concept and aspirations tend to have higher levels of ability (skill) and vice versa. Five of the highest correlated affective dimensions are self-related. There is also a tendency for students who hold that 'there is always a rule to follow' and there is 'little place for originality' in mathematics to perform well on tests. Although internationally standardized tests offered wide opportunities for advanced students to solve problems creatively, this may not have been reflected to a required extent in multiple-choice response coding. The latter

TABLE 8

Interrelations of single attitudinal statements to overall performance on mathematical tests

Hungarian secondary school students, $N \sim 2450$

Part 1,

Attitudinal statements about forms of mathematical activity

15 activity forms, 45 item statements

FORMS OF MATHEMATICAL ACTIVITY	Scale	Pearson's *r* (Correlation coefficient)
Highest positive correlations ($r > 0.25$)		
NONE		
Highest negative correlations ($r < -0.25$)		
Solving word problems	Preference	- 0.31
Solving word problems	Difficulty	- 0.31
Proving theorems	Preference	- 0.28
Differentiating functions	Importance	- 0.26

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TABLE 9

Interrelations of single attitudinal statements to overall performance on mathematical tests

Hungarian secondary school students, N ~ 2450

Part 2,
Attitudinal statements about mathematics
(as a subject and as a science)

46 attitudinal statements

STATEMENTS	Pearson's r (Correlation coefficient)
Highest positive correlations ($r > 0.25$)	
I would like to work at a job that lets me use mathematics	0.40
I usually understand what we are talking about in mathematics	0.36
I feel challenged when I am given a difficult mathematics problem	0.36
I usually feel calm when doing mathematics problems	0.31
I think mathematics is fun	0.30
There is always a rule to follow in solving a mathematics problem	0.28
There is little place for originality in solving mathematics	0.26
Highest negative correlations ($r < -0.25$)	
I am not so good at mathematics	- 0.35
I could never be a good mathematician	- 0.31
Mathematics is harder for me than for most persons	- 0.30
If I had my choice I would not learn any more mathematics	- 0.29
Mathematics is a set of rules	- 0.28
Now matter how hard I try I still do not do well in mathematics	- 0.27

two results are in good agreement with several observations concerning Hungarian mathematics teaching, pointing at the possibility that instruction does not really require the average student to be original and creative, and -- with exceptions -- encourages rule-following and conventionalism.

The negatively correlated statements are again mostly self--concept descriptors (statements 1st to 4th and 6th--7th in order of mention). It is puzzling at first sight that persons negating 'mathematics is a set of rules' likewise tend to reach higher ability levels. The possible explanation is twofold. First, one may be aware that mathematics involves much more than rule application without denying the importance of rule application in everyday school mathematics. Second, it is not inconceivable that the cited correlations stem from two partially different sets of cases. One may contain conventionalists who fare quite well in a traditional school context of mathematical problem solving, while the other, students who reach favourable levels of skill just because relying on problem-specific novel solutions, originality, or heuristics.

The first International Mathematics Study (HUSÉN, 1967) found little variance (among countries measured) in strength of corelations between affective variables and mathematical 'achievement'. Pearson r's varied from 0.26 (Belgium/Flemish) to 0.42 (Japan) with a mean of 0.32, with data based on nationally representative samples in all participating countries. Another representative study in the United States with large samples (CROSSWHITE, 1972) demonstrated a similarly strong correlation (0.28) which did not change much between age cohorts of 6th/8th grades and 9th/12th grades. These results and analyses on smaller Hungarian samples (KLEIN, 1971, 1973, 1975, 1977, 1980, 1987) confirmed that affective input characteristics significantly contribute to levels of mathematical achievement. What is more interesting at present is the finer structure of these affective variables (opinions) and their relation to more narrowly specified mathematical activities in school. Nevertheless, correlations reported in Tables 8 and 9 are of the same magnitude as ones between overall attitude score and achievement in international studies.

Attitudes of teachers toward mathematics

While students' attitudes toward, and opinions on, mathematics influence mathematical thinking and abilities, teachers' attitudes toward the same may play a decisive role in determining teacher behaviour and efficiency. The Second International Mathematics Study provides unique possibilities to compare teacher attitudes with student attitudes. The two domains of affectively coloured opinions listed above were re-measured with teachers of mathematics, albeit in a somewhat reduced fashion. The number of activity forms specified as attitude targets was smaller (and not fully overlapping) with student attitude measurement. The number of statements in the "general" part was also cut to fifteen.

The importance ratings of four common activity forms of school mathematics are shown in Table 10. The limit of interpretation being the same as for data in Table 4, all four specified activities seem to be judged 'very important' by teachers. Checking answers, solving word problems and memorizing rules are items we located among important forms of activity in students' perceptions as well. These four areas of mathematical practice reveal a congruence between student and teacher opinions. Table 11 includes selected data for Difficulty ratings. Of four activity forms, only one exceeded the limit: 'Word problems', which are evaluated 'very difficult' by teachers. This opinion is again identical with that of the majority of students (Table 5). Teachers express rather pronounced preference for three of the four activity areas (Table 12).

Although the self-related scales were not tested with teachers, results from the domain of affective dimensions at the "more general level" (Mathematics as a subject and as a science) have again some coinciding judgements between teachers and students. Using the same limit of interpretation as in Table 7, dimensions creating strongest agreement and disagreement among teachers are

TABLE 10

Attitudinal statements: Ratings of importance
Hungarian secondary school teachers, $N \sim 94$

1, most important,...5,least important
4 activity forms rated, 12 items

FORMS OF MATHEMATICAL ACTIVITY	$\bar{x}$
Rated most important ($\bar{x} < 2.5$)	
Checking an answer to a problem	1.44
Solving word problems	1.47
Memorizing rules and formulae	1.71
Estimating a result of a problem	1.82

TABLE 11

Attitudinal statements: Ratings of difficulty
Hungarian secondary school teachers, $N \sim 94$

1, easiest,.... 5, most difficult
4 activity forms rated, 12 items

FORMS OF MATHEMATICAL ACTIVITY	
Rated easiest ($\bar{x} < 2.5$)	
NONE	
Rated most difficult ($\bar{x} > 3.5$)	$\bar{x}$
Solving word problems	3.81

TABLE 12

Attitudinal statements: Ratings of preference
Hungarian secondary school teachers, $N \sim 94$

1, most liked,.... 5, least liked
4 activity forms rated, 12 items

FORMS OF MATHEMATICAL ACTIVITY	$\bar{x}$
Rated most liked ($\bar{x} < 2.5$)	
Solving word problems	1.89
Estimating a result of a problem	2.14
Checking an answer to a problem	2.18
Rated least liked ($\bar{x} > 3.5$)	
NONE	

listed in Table 13. Teachers widely hold that mathematics helps to 'think logically' and (with a similar semantic in the minds of most mathematics teachers) to 'think according to strict rules'. Creative aspects of mathematics are accepted (at least at a theoretical level) by many teachers. They hold that mathematics is a scene for creative persons and that there are many ways to solve problems. Assistance in acquiring logical thinking and alternative solutions to problems are opinions with which students and teachers strongly agree.

Teachers disagree with the assertion that 'mathematics is no more than a set of rules' and that it 'requires memorizing' mostly. Not surprisingly, they also judged the statement of 'little place for originality' in solving mathematical problems as not acceptable. These views harmonize with what was demonstrated (Table 7) among students concerning statement 'Mathematics involves mostly memorizing'. The ambiguity raised by students' accepting that problems 'cannot be solved' without using rules is, however, absent from teacher judgements. Mathematics teachers, unlike students, strongly deny that there have been no discoveries in modern mathematics.

CONCLUDING REMARKS

As achievement data prove there is no real ground to formulate general statements about secondary mathematics education in Hungary. There are strata of secondary schooling with excellent mathematics teaching and creativity-enhancing, innovative methods. On the other hand, more than half of the student population (almost all pupils in SVSS) suffer from grave problems in both achievement in, and attitudes toward, mathematics.

As a result of education, a majority of persons leaving secondary institutions hold that mathematics is important; that it is indispensable in an age of rapid technological progress. Another general belief is that a man-in-the-street is becoming less and

TABLE 13

Attitudinal statements about mathematics
(as a subject and as a science)

Hungarian secondary school teachers, N ~ 94

1, strongly disagreed with ....
5, strongly agreed with

STATEMENTS	
Agreement strongest ($\bar{x} > 3,75$)	$\bar{x}$
Mathematics helps one to think logically	4.68
Mathematics is a good field for creative people	4.26
Estimating is an important mathematics skill	4.04
Mathematics helps one to think according to strict rules	3.98
There are many different ways to solve most mathematics problems	3.80
Disagreement strongest ($\bar{x} < 2,25$)	
Mathematics is a set of rules	1.69
Learning mathematics involves mostly memorizing	1.77
There have not been any new discoveries in mathematics for a long time	1.86
There is little place for originality in solving mathematics problems	1.96

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less competent in the field. Students do not blame specific methods and processes of mathematics encountered in school for their felt incompetence. They do not even question the declared necessity of learning mathematics in higher grades of secondary schooling in case of persons who do *not* want to be employed in jobs with mathematical requirements. This abstract assignment of importance is compatible with the view that it is another person, not the self, for whom mathematics is useful. It is also compatible with a *dispreference* -- especially in pupils of no personal interest in mathematics -- toward related activities in school. Persons not interested in mathematics typically profit very little from taking secondary mathematics. The time is certainly drawing near when larger strata of pupils will express *doubts* about compulsory mathematics imposed upon them. As it is today, importance attributed to mathematics inside and outside the school system is overestimated. Educators and the society at large still presuppose that personal mathematical achievement at secondary level testifies highly developed thinking skill or even creativity. That is not borne out by nationally representative studies. A decline in perceived importance will surely follow if we make no steps to modernize content and process of secondary mathematical education.

The key of transformation is, obviously, the teacher. Hungarian secondary mathematics teachers being overstressed, mostly inadequately trained and underpaid, there are obstacles in the way of modernization. The professional setting and job context of teachers should be modified appreciably if we want teachers to volunteer self-training, improve their classroom methods, and implant a liking of mathematics into their students.

NOTES

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3/ The acronym 'O.P.I.' stands for 'Orszagos Pedagogiai Intézet', Hungarian name of the National Institute of Education where Professor Varga worked out his method.

4/ The authors are greatly indebted to the International Association for the Evaluation of Educational Achievement for the opportunity of using measuring instruments and connected research materials. We are especially grateful for the help of Professors T.N. Postlethwaite, R.W. Phillips, R.A. Garden, and of Mrs. Julianna Radnai-Szendrei (N.R.C.).

The second author acted as co-ordinator of the Population B (cross-sectional) measurement in Hungary for the SIMS. The results used as illustrations in this paper are taken from simplest descriptive and correlational studies performed on the RMPP data system. For a fuller report of results from this system see HABERMANN (1983, 1985, 1986, 1987; RADNAI-SZENDREI and HABERMANN, 1984).

5/ It should be underlined that although numerical results here are fairly objective the interpretations are those of the authors only. Several alternative interpretations were voiced in other Hungarian publications on the subject.

6/ The issue of semantics of generic terms in psychology cannot be discussed in this practical paper. As pointed out elsewhere there is more agreement in interpreting differential targets of affective constructs than inherent conceptual characteristics of these constructs. For instance, it is easier to distinguish preferences toward mathematics from preferences toward physics than preferences toward mathematics from interests in mathematics.

7/ In his later work, J.B. Carroll accepted the view that perseverance as a single construct is not sufficient to describe the full range of school subject specific affective / motivational components (cf. CARROLL, 1984/1985 edn., p.93)

8/ These variables were measured by such instruments as Dutton's 'Attitude toward Mathematics Scale' (RYAN, 1969), 'Elementary Attitude Scale toward Mathematics' (ANTTONEN, 1969), 'Secondary Mathematics Attitude Scale' (ANTTONEN, 1969) and the 'Pro-Math Composite' instrument (CROSSWHITE, 1972).

9/ As done for teachers, it may be useful to demonstrate by a set of distribution characteristics the span of the student sample. The age of pupils varied from 17:3 to 20:7, with the average 18:1. 62 % were girls and 38 % boys. Among fathers, less than 1 % were unemployed. 14 % were unskilled workers, 45 % skilled workers, small craftsmen, low-level white collar and clerical employees, 15 % foremen and low--level managers, 9 % middle-level managers and directors, 11 % higher level managers and directors. The monthly full income of families (reported) at the time of measurement was HUF 3000,--or lower in 4 %, 3001--4000,-- in 7 %, 4001--5000 in 11 %, 5001--6000 in 19 %, more than 6000,-- in 58 % of the sample. 23 % of the fathers had secondary and 23 % of them, university or higher college certificates. The same proportions for mothers were 28 % and 11 %. 7 % of fathers and 9 % of mothers did not even complete 8-grade compulsory Hungarian elementary school.

10/ This description is only meant to distinguish a second set of affective rating items from the 'activity specific' first set.

REFERENCES

Anttonen,R.G.,
Longitudinal study in mathematics attitude.
Journal of Educational Research, 1969 (*62*).

BLOOM, B.S.,
Human characteristics and school learning.
New York (etc.), 1976: McGraw-Hill Book Co.

Brookover,W.B., Shailer,T., Paterson,A.,
A self-concept of ability and school achievement.
Sociology of Education, 1964 (*37*), pp.271--278.

Carroll,J.B.,
A model of school learning.
Teachers'College Record, 1963 (*64*), pp.723-733.

Carroll, J.B.,
The model of school learning: Progress of an idea.
In: Fisher,C., Berliner,D., Perspectives on instructional time.
New York (etc), 1984: Longman Inc.
/Edition used: Anderson,L.W., ed., Perspectives on school learning. Hillsdale,N.J., 1985: Lawrence Erlbaum. pp. 82--108./

Crosswhite,F.J.,
Correlates of attitudes toward mathematics.
In: Wilson,J.W., Begle,E.G., eds., National longitudinal study in mathematics achievement. Report 5/20.
Palo Alto, Calif., 1972: School Mathematics Study Group.

Dutton,W.H.,
Measuring attitudes toward arithmetic.
The Elementary School Journal, 1954.

Dutton,W.H., Blum,M.P.,
The measurement of attitudes toward arithmetic with a Likert--type test.
The Elementary School Journal, 1968.

Farquhar,W.W., Christensen,E.W.,
Academic achievements of eleventh grade Puerto Rican high school students.
East Lansing,Mich., 1963: Office of Research and Publications, College of Education, Michigan State University.
(Educational Publication Series, no.43)

Habermann,M.G.,
A multivariate educational-psychological study on a nationally representative sample of 4th grade secondary school pupils.
(In Hungarian)
Neveléselmélet és Iskolakutatás, 1983(II), no.2, pp. 5--92.
(=Part 1); ibid.,no.3,pp.5--93. (=Part 2)

Habermann,M.G.,
The system of variables in the nationally representative multivariate educational-psychological investigation of secondary school students.
(In Hungarian)
Neveléselmélet és Iskolakutatás, 1985(IV),no.2,pp.40--115.

Habermann,M.G.,
Environmental variables as determinants of the comprehension of interpropositional relations in grammar school and specialized vocational school students.
(In Hungarian)
Neveléselmélet és Iskolakutatás, 1986(V), no.4,pp.83--150.

Habermann,M.G.,
Higher order structures in abilities of comprehending sentences containing complement-governing predicates.
(In Hungarian)
In: Barkóczi,I. (compiler), Magyar Pszichológiai Társaság VIII. Országos Tudományos Konferenciája. Budapest, 1987: MPT. Volume I. p. 21.

Helmke,A.,
Mediating processes between children's self-concept of ability and mathematics achievement: A longitudinal study.
München, 1987: Max-Planck-Institut für psychologische Forschung. (Series "Papers", no.6)

Helmke,A.,
The influence of students' task motivation and self-concept of aptitude on different indicators of effort at school.
International Journal of Educational Research, 1988(in print)

Kifer,E.,
Second Study of Mathematics. Report of the attitude trials.
(place not indicated), 1979: I.E.A./University of Kentucky.

Klein,S.,
Investigation of the psychological effects of the 'composite' method of teaching mathematics.
(In Hungarian)
Magyar Pszichológiai Szemle, 1972(XXIX),pp.12--20.

Klein,S.,
The New York Study. A comparative evaluation of two Dienes Projects: École Eymard(...) versus Fleming School(...)
Sherbrooke (Canada), 1973: Centre de Recherche en Psycho-mathématique, Université de Sherbrooke.

Klein, S.,
The evaluation of the O.P.I. Mathematics Project.
In: Contributi originali dell' I.S G.M.L.,
Firenze, 1975: Edizioni O.S. pp. 5--76.

Klein,S.,
On psychological problems of mathematics education.
(In Hungarian)
Pedagógiai Szemle, 1977 (XXVII), pp. 327--338.

Klein, S.,
A psychological study of effects of the 'composite' method of mathematics education.
(In Hungarian)
Budapest, 1980: Akadémiai Kiadó. (351 pp)

Klein,S.,
The effects of new mathematics.
Budapest, 1987: Publishing House of the Hungarian Academy of Sciences.

Maertens, N.,
Effects of arithmetic homework upon the attitudes of third-grade pupils toward certain school-related structures.
School Science and Mathematics, 1968(68), pp. 659-662.

Radnai-Szendrei,J., Habermann,M.G.,
Intercurricular differences in the mathematics achievement of 4th grade secondary school students.
(In Hungarian)
Pedagógiai Szemle, 1984(XXXIV), pp. 130-143.

Ryan,J.J.,
Effects of experimental programs in mathematics on relevant attitudes and interests of ninth grade pupils, as measured by questionnaire indices.
St.Paul,Minn., 1967: Minnesota State Department of Education.
(Interim Report Project no.5/1028).

Ryan,J.J.,
Effects of modern and conventional math curricula on pupil attitudes, interests, and perception of proficiency.
Washington, D.C., 1969: United States Office of Education (Bureau of Research).

Varga,T.,
Report of the Budapest Center.
Bulletin of the International Study Group for Mathematics Learning, 1962 (I), pp. 3--4.

Varga, T.,
Pilot work in Budapest.
Bulletin of the I.S.G.M.L.(see above), 1964 (II), pp. 1--5.

Varga, T.,
The use of a composite method for the mathematical education of young children.
Bulletin of the I.S.G.M.L.(see above), 1965 (II), pp. 1--9.

Varga, T.,
Higher mathematics in lower grades.
In: Feldman,L., ed., Prepubescent mathematical learning.
San José, Calif. , 1967: San José State College. pp. 18--22.

Varga, T.,
Toward an up-to-date teaching of mathematics.
(In Hungarian)
In: Forrai, T.,et al., eds., Néhány hazai és külföldi kísérlet.
Budapest, 1972: Tankönyvkiadó.

Varga,T.,
Post-"new math" since 1963: Its implementation as a historical process.
In: Wirszup,I., Streit,R., eds., Developments in school mathematics education around the world. Chicago, 1987: Council of Teachers of Mathematics. pp. 237--249.

Additional item

Husén, T.,
International study of achievement in mathematics: A comparison of twelve countries.
Stockholm/New York, 1967: Almqvist och Wiksell, Wiley and Sons.
/Vols. I, II./

BEYOND CONSTRUCTIVISM

(Learning Mathematics at School)

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Last year we heard plenary presentations that made all of us think more seriously about how we view math education in a wider context of epistemological and learning theories. The main focus, last year was on constructivism. I thought that research on the psychological aspects of math education studied in the last decade had their own contribution going beyond constructivism. This will be my main topic today. I would like to elaborate on three main issues. I will start with the role and characteristics of learning at school; I will then sketch some outlines of the epistemology for the mathematics learned at school and, finally, with an illustration taken from a specific piece of research I will speak on the role that misconceptions and planned environments can play in schools.

The role of schooling

My first assumption is that schools exist mainly and primarily to promote knowledge. Most of us accept the constructivist approach to the acquisition of knowledge, and I will quote Sinclair from last year, who said that "the essential way of knowing the real world is not directly through our senses, but first and foremost through our actions. ...new knowledge is constructed from the changes or transformations the subject introduces in the knower-known relationship.. (and) the quality of the knowledge gathered in this way is partly determined by the ways in which reality reacts to our interventions and by its correspondence to the knowledge other people have constructed" (Sinclair, 1987).

I would like to regard school as a significant component in the child's reality and would like to analyze some of school characteristics as an environment within which the child can exercise his theories in action. School itself can be looked at from various aspects: each one of them might contribute different sets of experiences. I will concentrate on school as an environment for learning mathematics, and will take into account the specifics of mathematics as well as the modes of learning and construction of knowledge.

In speaking of learning at school, I do not intend to underestimate the extent of learning which takes place outside school, rather to emphasize the role of schools in designing a learning environment. We all know that a lot of learning is going on outside schools. We are told that much of the mathematics that children and adults know and use is learned on the streets and in factories, and it is based not on one's early schooling but on one's experiences, actions and reflections at various sites (Reed & Lave, 1981). This is the story about the vendlers in Brazil that Carraher and Schliemann describe (Carraher, carraher and Schliemann, 1987); the fishermen in north Brazil (Schliemann, 1988); or the dairy workers that Scribner (Scribner, 1984) tells about them. Certainly, we admit the role of learning via free play outside school.

If this is the case, it might be helpful to understand the difference between learning in school and that outside school. Since Lauren Resnick (1987) recently made such an analysis I will use this as my point of departure. Resnick has "identified four general classes of discontinuity between learning in school and the nature of cognitive activity outside school. Briefly, (she said) schooling focuses on the individual's performance, whereas out-of-school mental work is often socially shared. School aims to foster unaided thought, whereas mental work outside school usually involves cognitive tools. School cultivates symbolic thinking, whereas mental activity outside school engages directly with objects and situations. Finally, schooling aims to teach general skills and knowledge, whereas situation-specific competence dominates outside" (Resnick, 1987, p.16).

I am not sure these characteristics tell us the entire story. I think that there is another aspect that she neglected to mention; this has to do with the fact that school is an environment purposely and intentionally created to promote knowledge (and in promoting knowledge I include norms and social knowledge as well). Instruction at school is a goal-directed, intentional, and conscious activity on the part of schools, and therefore amenable to rational analysis and critical consideration.

Learning in and out of school has a completely different setup. Learning out of school is part of the immediate social and economic system. The goal on the part of the trainer is to put the trainee as soon as possible on the production line as far as a skill is concerned, or, to improve other skills, as far as social communication is involved. This is not exactly the case at school. Schools aim to pass on knowledge to students to partly be employed at school in further learning, but mainly to be employed elsewhere, after leaving school. This creates quite an awkward situation in comparison to the rest of the world. Schools do not gain directly from the students' knowledge but rather from the growth in their teachers' knowledge of how to teach. Pedagogy, and means of supporting learning, are the expertise of schools. This is not the situation outside school. The carpenter or the computer-scientist are the experts for carpenting or computing, respectively. They are not the experts of how this kind of knowledge is best learned

Such an observation raises several fundamental questions which are related to the fact that schools have to deal with questions of motivation or with questions of rewarding procedures, etc. These will not concern me here. They are all relevant to learning. Yet, admitting the limitation in scope, I will concentrate here on the cognitive activity, on the learning of conceptual systems which are at the heart of schooling. In what follows, instead of speaking in general terms I will refer directly to the learning of mathematics. The comparison between learning mathematics in and outside school raises the questions: What kind of mathematics do we teach in school? For what purpose? And, how do we teach it in the light of the constructivist maxim?

Obviously we do not teach addition in the context of a supermarket with the goal of saving some money. At most, we mimic such a situation by what is called "word problems". This has no pretention of being any

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reality in the learner's eyes. We do not teach graphs and diagrams in the context of advertizing and check what is the best way to convey information and what is the preference of each type of diagram from the advertizing point of view. We do not teach ratio in the bakery, though many activities there call for the use of ratio and proportion. And finally when or where should we teach Euclidian geometry, vector space or, exponential equations?

The utilitarian approach with its immediate payoff which is typical of learning out of school becomes very weak even as a point of departure when it comes to school. Of course, there is also a payoff in learning mathematics at school. Starting with the jobs available for one on his completing school, but also as a means of survival in schools while attending it (in the western society), or in the process of learning itself. I think, nevertheless, that mathematics learned at school has a completely different agenda than the utilitarian one. I suggest that learning mathematics at school is aimed at learning a specific conceptual framework as a cultural endowment that shares some of the characteristics that Resnick mentioned, such as being abstract, general, symbolic, and detached from a specific context.

In order to make my argument, I would now like to raise some epistemological considerations related to the learning of mathematics. The discussion that follows should not be regarded as a philosophical attempt to resolve the controversy between the different schools within the philosophy of mathematics (Kitcher, 1983; Benacerraf and Putnam (Eds.) 1985)), rather some of the assumptions mentioned here will be used to clarify the educational issues. Different approaches to the teaching of arithmetic correspond to different conceptions about the reference and the sense of the mathematical language.

The theoretical approach upon which I will elaborate here addresses two main pedagogical needs (a) the need for any learner to construct his knowledge through interaction with the environment (the constructivist maxim), and (b) the need to arrive at mathematical truths (the realist maxim). I know that the second need is not acceptable to some constructivists, but I contend that my assumptions reside within the non-radical constructivism. To avoid ideological dispute I am ready to name my approach "a pedagogical realism". The epistemology that I call "pedagogical realism" has grown out of considering three issues: (1) the ontological status of mathematical entities; (2) where can the learner look for the truths of his mathematical findings and beliefs; and (3) what are the pedagogical implications from the above for learning mathematics at schools.

The Ontological Question

I begin with analysis of natural numbers, since those seem to be the most confusing in regard to the ontological question. Clarifying their nature will make it easier to understand other kinds of numbers (such as rational, decimal, irrational) or other mathematical entities. In discussing the question, "What are numbers?" I first make the distinction between the linguistic signs and what they stand for (symbolize or signify). Furthermore, I make the distinction between number words which are part of ordinary language, and numerals which are part of the symbolic language of mathematics. People usually think that both refer to the same entity. But number words in ordinary language are used to qualify quantitatively other objects which are specified by their class term, such as "children" in the case of "five children". Numerals in mathematics refer to mathematical entities, abstract concepts or objects. The numeral "5" for example is a name for a specific and unique number that has all kinds of properties now to be learned at school, such as being an odd number, or that it is one of the square roots of 25, etc., all of which are concepts within the mathematical system. Though we frequently think of the number 5 as of having an image of multitude of elements, in the language of mathematics one speaks of the number 5 in the singular, e.g. "5 is an odd number", a point that emphasizes our conception of it as a unique and singular mathematical object that has an ontological status in the language of mathematics that differs from the status of "five" as a quantifier of other objects in ordinary language.

Obviously, the young child starts to conceive numbers in their relation to other objects as they are used in ordinary language, but later on at school we aim to teach him that numbers within the mathematical system refer to abstract mathematical entities. In his spontaneous environment the child will hear expressions such as: "John ate five cookies", or "John is five years old", in school he will soon deal with expressions of the form "5+3=8" or "5 is a prime number". The distinction I have just drawn is even sharper in moving toward mathematical entities, that do not have seemingly parallel expression in everyday experience and in ordinary language such as the number "e" or "pi" or concepts such as polynoms, or vectors. What creatures are they?

Mathematics is a conceptual system with its own unique language. Its language enables us to look for its sense and reference which are characteristics of any language. For example, the following expressions: "2+2"; "2x2"£ "2 to the power of 2" all have the same reference which is the abstract number 4. Yet each expresses a different thought and a different sense. The different senses of the symbolic language of mathematics comprise its essence and the gist of understanding mathematics. Most senses expressed in mathematics are new relations that could not be expressed in ordinary language and now are embedded in the new symbolic language to express notions which are impossible to express otherwise. The most trivial example is the attempt to express in ordinary language the algebraic expression: "three times of a number increased by 7". This expression is ambiguous as it stands, and only the symbolic language of mathematics can remove the ambiguity by expressing this either as 3x+7 or as: 3(x+7).

The Truth of Mathematical Sentences

I have raised the ontological question since it is intimately connected to the question of truth in mathematics. It is expected that children will learn the rules of formation of mathematical sentences at school, that they will understand the different senses of each sentence and that they will know how to distinguish between true and false statements in this language. Moreover, much of our teaching in mathematics pertain to the discovery of new expressions that maintain the truth value of the sentence. It should be noted (as Russell wrote in 1959¥1912 p.70) that having a truth value is a property of beliefs, yet it is established by many different methods, which are independent of the beliefs (As the long history of false beliefs has demonstrated). Also, falsehood is adjunct to the notion of truth. In Russell's words: "Our theory of truth must be such to admit of its opposite, falsehood" (Ibid). We might call it "false belief", "error", "misconception", "bug" and in some cases even "theory in action", all express a system of beliefs that has a truth-value with its two complementary values: true and false.

One method of arriving at truths in mathematics is what Russell called the coherence theory of truth. Actually, all proofs in mathematics are made in a deductive manner to preserve the coherence of the system. The trouble is that young children (and many adults!) that study mathematics, cannot prove for themselves the truths of mathematics, by means of the deductive method.

A real and significant educational question is, if the deductive method is ruled out for young children, how can a child know that he has arrived at a true sentence. Or how can he know which numeral, for example, to put in the blank space of the expression 2+ =5 and get true sentence?. The answer that we seek has nothing to do with the syntactic rules but rather with knowledge about abstract numbers and their characteristics. To write 2+3=6, is syntactically as good as writing 2+3=5. Yet we and soon every child will know that the first statement is false and that the second is true. How should they know it? The paper on which the child writes a statement such as: 3+2=10 is long-suffering and there is nothing in this pen and paper activity to tell the child whether he is right or wrong.

The Pedagogical Outlet

If the deductive method is ruled out for the young child for arriving at true sentences, we are left with two other options: One is to learn and memorize the true facts that expert mathematicians know how to prove, and later on to refer to the memory as the judge of the truth value of a given sentence (or rely on teachers' approval or disapproval). A second alternative is to provide the child with some tools for verification via experimentation in a world assumed to be analogous to the mathematical abstract world; in which he already knows what is true and what is false there. Logo (Papert, 1980) could be such an example. While the child operates the turtle by means of formal language, he gets his feedback about the truth value of his formal descriptions by comparing the move of the turtle to the goal he had in mind and he knows (eventhough he will not always admit it), whether he succeeded in his formal description or not.

Each of the above approaches has a catch in it. The first approach (approaching the teacher for an approval) relies heavily on authoritarian methods of imposing knowledge and discourages self-exploration and self-conviction. In fact, many teachers have taken on themselves this role of judging whether the child is correct or not in his performance. This does not let the child construct for himself the mathematical notions and concepts. Nor does it enable him to realize that the truths of mathematics are objective and necessary.
The second approach of experimentation as the mode for verification introduces mathematics as an empirical science rather than a deductive one. In adopting the second approach we adopt also another theory of truth, the correspondence theory of truth (Tarski, 1949, Russell,

1959/1912) by which truth consists in some form of correspondence between a belief and a fact. If to use Tarski's notorious example: "The sentence "snow is white" is true if, and only if, snow is white".

Using a similar approach to learn the truths of mathematical sentences means constructing a world in which the child will be able to examine for himself the truth of mathematical sentences via the state of events in a familiar world. It seems that the accumulated wisdom of math educators agree that learning mathematics at school can benefit from designing such environments in which children can experiment; get feedback on their activities; be able to explore their hypotheses and discover whether they are true or false. Note, again, how crucial is the notion of truth and falsehood to the examination of one's hypotheses.

At this point some of you might think that I am speaking about microworlds, and indeed I am. But I would like to spell out what makes a microworld such a powerful educational tool. Elsewhere I have talked about the characteristics of a learning environment at school (Nesher, 1988) and have detailed what I call Learning Systems (or microworlds if you wish). Here I will point only to some of its necessary components.

The most important characteristic of a Learning System is that it consists of two major components:

1) There is a clear articulation of a unit of knowledge to be taught, based on an expert knowledge. The experts in this case are not the scientists in the field, but rather the experts who can tailor the

body of knowledge to the learner's particular constraints (such as age or ability).

2) There is an exemplification component that must be familiar to the learner. The child should be able to grasp intuitively truths within the selected exemplification model. The choice should also ensure that the relations and operations among the objects in the exemplification component will fully correspond in an isomorphic manner to the objects relations and operations to be learned. I should stress that the exemplification component must be familiar to the child to serve as an anchor to develop an understanding of a new and unknown system of relations. For example, one can use the ability of the child to differentiate among colors and lengths as is done in the use of Cuisenaire rods, to teach him new concepts related to the natural numbers, or one can use the child's knowledge of spatial relations to learn programming in Logo.

What is possible and what is not possible within such an environment? It seems that it is possible to establish a language that employs ordinary language notions between the teacher and the child about the objects and relations in the exemplification model. It is possible to demonstrate new configurations not yet experienced by the child, in a domain which is already familiar to him. It is possible to try and have a language that symbolizes these new configurations, but it is not possible to assume (and this is a warning) that, just by pointing out to him, the child has instantly noticed the new relations, has reflected on them, and has absorbed their significance. Nor should one assume that the language the teacher uses to signify these new relations, is not an empty verbalization for the child. Here, I call again upon the constructivist maxim.

What is really possible is to construct within the child's reach an environment that if experienced and explored by the child has the potential of revealing some interesting relations that were (or were not) noticed before by mathematicians and were coined in a mathematical language. In offering such learning systems there is a notion of planning and goal-directed activity at school which is the essence of instruction, as part of our vision of mathematics as a cultural endowment, on one hand, and an agreement that the learner must construct for himself any piece of knowledge, on the other.

Granted that mathematical knowledge grows out of the subject's reflection on his own actions, the best we can do is to build a constrained environment within which the child can act freely in a less scattered and random fashion than might happen outside school. My claim is that most of the more advanced mathematical notions are not easily amenable to the student's activity or experimentation outside of schools and as such they call for some pedagogical intervention.

In advocating the planning and creation of learning systems by the schools, a word of caution is in order. There is a danger that there will be a gap between teachers' planning and the child's necessity to construct for himself his knowledge at his own rate.
This is a true pedagogical dilemma with which teachers have to live every day. When a teacher has his own expertise and norms, then the students' theories in actions are doomed to be judged relative to the

norm as misconceptions. I would like therefore to raise a question, on which it is not always clear what the radical constructivists stand is: Does such a knowledge on the part of the teacher enhance or inhibit learning? I would like to answer this question after presenting the following research findings.

Learning Decimals

In the past few years some of my students have been engaged in research about the learning of decimal numbers. I will use these studies to illustrate the problems of learning at school. The first study by Mrs Bilha Zuker, my former M.A. student, dealt with the question of understanding vs. algorithmic performance in the domain of decimal fractions. She constructed two separate tests, one that tested what we believed to be routine algorithms involving decimals and the other test that called for some reflection and understanding of decimals. Comparing the value of two decimals was one of the items in the second test.

The next step was taken by Dr. Irit Peled who wanted to look more closely where the failure in understanding occurred. She asked herself if it would be possible to find its sources. Her study was parallel to a study made by Leonard and Grisvald (1981) and Resnick et al (1988). To cut a long story short, I will describe the gist of the study. Consider for example the following tasks administered to children of grades 6,7,8, and 9. The subjects had to mark the larger number in the following pairs:

Case I	0.4	vs.	0.234
Case II	o.4	vs.	0.675

In Case I Jeremy marked that 0.234 is larger than 0.4; and in Case II he marked that 0.675 is the larger one. Does he or does he not know the order of decimal numbers? In our study in Israel the data was gathered in individual interviews, so that the children could explain their choices. This helped us understand their theories in action. In both cases Jeremy said that the longer number, i.e. with the more number of digits (after the decimal point) is the larger number (in value). Jeremy had one guiding principle as to the order of decimals and, accordingly, Jeremy was wrong in case I while in Case II he was right. Although his guiding principle was a mistaken one, he succeeded in correctly solving all the exercises similar to Case II. Also, It is not hard to see that his guiding principle was one that served him well up to this point, having been extrapolated from his knowledge of whole numbers where the longer numbers really are larger in value. And, unless something is done, Jeremy's "success" or "failure" in certain tasks is going to depend on the actual pair of numbers given to him. This, of course, blurs the picture of his knowledge Now imagine Ruth who decided in both Cases I and II (in the above example) that 0.4 is the larger number, i.e., in each case she pointed to the shorter number as the larger one in value. Ruth gave the following explanation: "Tenths are bigger than thousandths, therefore, the shorter number that has only tenths is the larger one." Ruth does not differentiate between Case I and Case II either. She will be correct in all the cases similar to Case I, but wrong in all cases similar to Case II. We can understand this kind of reasoning in light of what is learned in fractions. Ruth has a partial knowledge of ordinary fractions and cannot integrate what she knows about them with the new chapter on decimals fractions and their notation. It is interesting to note that about 35% of the sixth graders in Israel who completed the chapter on decimals acted like Jeremy. They were, in fact, using the above-mentioned rule which relies heavily on the knowledge of whole numbers; and about 34% of the Israeli sample of sixth graders made Ruth's type of rule. Even more interesting, is the fact that while Jeremy's rule frequency declines in higher grades, Ruth's rule is more persistent and about 20% of the seventh and eighth graders still maintain the rule laid out by Ruth (Nesher and Peled, 1984)

How could such rules persist for such a long time? One reason is that in most cases this topic is learned only in a formal mode as a pen and pencil exercise. The child cannot find for himself the truth of his sentences and must rely on what I called the first option, i.e. that the teacher tells him whether he is right or wrong. Soon I will show that the teacher, too, in this case could not be of great help.

In many cases, as we saw, the mistaken rule is disguised by a "correct" answer. That is, the student may get the "right" answer for the wrong reasons. Thus, for the student who holds a certain rule, not all the exercises consisting of pairs of decimal numbers will elicit an incorrect answer. For example, decimals with the same number of digits are compared as if they were whole numbers and, therefore, these items are usually answered correctly even by those who hold inadequate rules, and cannot therefore be used for diagnosing or raising conflicts with mistaken rules. This was the case with the previous example. If the student for example, given the following item "Which is the larger of the two decimals 0.4 and 0.234?" answers 0.234

we may suspect that he holds Jermey's rule. But, if he answers 0.4, we cannot know whether he knows how to order decimals, or if he is holding Ruth's error, but happened to get lucky numbers and be correct on this particular item. Thus this item can discriminate and elicit those holding Jeremy's rule but cannot discriminate between those holding Ruth's rule and those who really know the domain. Along these lines, for the same task, the pair of numbers 0.4 and 0.675 can discriminate those holding Ruth's rule but cannot discriminate between those holding Jeremy's rule and experts.

I have mentioned before that the teachers could be but of little help to their students even if they were ready to take an authoritarian stand. The reason is that these misconceptions are hard to detect. In her work Irit Peled has built a simulation that produces pairs of decimals to be compared. In Peled's simulations it was found that when pairs of numbers are randomly selected from all the possible pairs of numbers having at most three digits after the decimal point, the probability of getting an item that will discriminate Jeremy's rule was 0.10, and Ruth's rule 0.02. Thus both Jermey and Ruth will succeed up to 90% on any activity given to them if a special consideration is not taken into account. It is not surprising, then, that teachers are usually satisfied with the performance of children holding Jeremy's or Ruth's rules. And for this they should not be blamed. On the basis of one wrong item it is impossible to discover the nature of the student's theories in action. In such classroom it will also be very difficult for Jeremy and Ruth to give up their rules since they are daily rewarded for their erroneous rules by correctly answering non-discriminating items.

If we think that students experience at school should aid in constructing more elaborated rules than the ones currently held, than their experience should be directed towards only discriminating items that raise conflicts with their earlier notions. Moreover, they must have the opportunity to realize that there is a conflict. This cannot to be achieved by the teachers feedback, for whom the students rules are sometimes masked. We must look for a learning environment and learning systems that can exemplify to the student what is right and what is wrong from a mathematical expert point of view.

Why did I take up your time in describing this detailed example? Because the proceedings of our meetings in the last decade are full of papers dealing with similar examples which are, in my view, the seeds for the theory needed for a more successful teaching of mathematics in schools.

I, for example, have learned several lessons from the above example which I will present in general terms:

(a) Misconceptions and erroneous rules are found not only behind erroneous performance, but also lurking behind many cases of correct performance. They are often "masked" by correct performances.

(b) Misconceptions may persist if the children do not have the opportunity to experiment directly with a reality that contradicts their beliefs.

(c) Schools are for creating learning environments that have their own feedback mechanisms, and for constructing good diagnostic discriminating items for the child's activity. Teachers are not in schools to offer feedback, or for testing and rewarding. Rather, on the basis of their past experience they should be the experts on what are the best activities that enhance children's learning. Watching classes I am surprised (and you are, too, probably) how much the child works on exercises that do not teach him anything.

Here I have touched upon the question that I have raised before, and one that I think constructivism does not explicitly answer. Granted that the child has to construct his own knowledge, what is the role of schools? Can schools be more efficient than any other occasional learning environment? I think, my answer is clear and I will summarize it by the line of thought that I have developed here:

I think that schools have the potential of becoming a better environments for learning mathematics than any other environment. It depends on our understanding of the nature of constructing mathematical knowledge. Understanding in this context means building a detailed theory that will first be based on a sound epistemology, and then will be sufficiently elaborated to become a theory that directs the actual learning; creating learning systems will be part of it; it will explain "the theories in action" that children develop as well as discern discriminating items and tasks that many of us work on.

Final note

None of the hopes for good schooling that I have entertained must be realized. None of the dreams about planned environments must be fulfilled. Currently, many schools actually demonstrated that I am wrong. Yet I believe it is time to sketch the potentials for better schooling and how we can reach them. I suggest that human creativity, and man-made learning environments should be developed according to our best pedagogical knowledge. I think that the so-called "artificial" environments need not be inferior in promoting knowledge than the natural ones, that we all know how powerful they are. Are man-made materials, such as the synthetic polymers, worst than the materials that Mother Nature supplied us with in the first place? Can any one imagine our world today without the advancement in all technologies which are man-made? And why should this also not be true of man-made learning environments that exemplify conceptual frameworks which are, themselves, a human creation? I believe that it is possible and it is in our hands to turn such dreams into reality.

References

Benacerraf, P., & Putnam, H. (Eds) (1985). Philosophy of Mathematics, (2nd Edition). Cambridge University Press.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D.(1985). Mathematics in the streets and in the schools. British Journal of Developmental Psychology, 3 (1), 21-29.

Kitcher, P. (1983) The Nature of Mathematical Knowledge. Oxford University Press.

Leonard, F., & Sackur-Grisvald, C. (1981). Sur deux regles implicites utilisees dans la comparaison des nombres decimaux positifs. In Bulletin de l'APMEP, 327, 47-60.

Nesher, P (in press). Microworlds in mathematical education: a pedagogical realism. In L. B. Resnick (ed.). Knowing and learnirg: Issues for a Cognitive Science of Instruction. Hillsdale, NJ: Lawrence Erlbaum Associates.

Nesher, P. & Peled, I. (1984). The derivation of mal-rules in the process of learning. In Science Teaching in Israel: Origins, Development and Achievements. Pp. 325-336. The Israeli Center for Science Teaching, Jerusalem.

Papert, S. (1980) Mindstorms Children, Computers, and Powerful Ideas, N.Y.: Basic Books, Inc. Publishers.

Peled, I. (1986). Development of the decimal fraction concept. A Ph.D Thesis, submitted to the Senate of the Technion - Israel Institute of Technology, Haifa.

Reed, H. J., & Lave, J.(1981). Arithmetic as a tool for investigating relations between culture and cognition. In Casson W. R. (Ed), Language, culture, and cognition. (pp. 437-456). New York: Macmillan Publishing Co., Inc..

Resnick, L. B.(1987). Learning in school and out. Educational Researcher, 16, 13-21.

Resnick, L. B., & Nesher, P. (1983). Learning complex concepts: The case of decimal fractions. Paper presented at the Twenty-Fourth Annual Meeting of the Psychonomics Society, San Diego, CA.

Resnick,l. b., Nesher, P., Leonard, F., Magone, M., Omanson, S., and Peled, I. (In press). Syntactic and semantic processes in concepts: the case of decimal fractions. Journal for Research in Mathematics Education.

Russell, B. (1912). The Problems of Philosophy. Oxford University Press.

Scribner, S.(1984). Studying working intelligence. In B. Rogoff & J. Lave (Eds), Everyday cognition: Its development in social context. (pp. 9-40). Cambridge, MA: Harvard University Press.

Sinclair, H. (1987). Constructivism and the psycholgy of mathematics. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds). Psychology of Mathematics Education. Proceedings of the Eleventh International Conference of PME. Montreal.

Tarski, A. (1949) The semantic Conception of truth, in Feigl, H. & Sellars, W. (Eds.) Reading in Philosophical Analysis, Appelton-Century-Crofts, Inc. N.Y. pp 52-85.

Reconstructive Learning

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Résumé:

Dans cette contribution,intitulée 'Un apprentissage reconstructif', on s'interroge sur les conditions nécessaires pour mettre les enfants dans une situation dans laquelle ils engendrent eux-mêmes- éventuellement avec l'aide de l'enseignant- des relations mathématiques abstraites.

En vue de répondre à cette interrogation, on porte l'attention sur des constructions et des productions fournies par des enfants dans le cadre d'un enseignement de mathématiques réalistes.

Entrent alors successivement en consideration les points suivants:

-Que peut bien signifier "les productions personnelles des enfants" et quelles sont leur fonctions dans le processus d'apprentissage? Exemples.

-Qu' apporte une recherche en didactique à l'égard d'un enseignement des mathématiques qui soit reconstructible?Quel rôle jouent les productions personnelles de ce point de vue?

-Interviennent enfin les caractéristiques d'un enseignement des mathematiques réalistes.Ces caractéristiques determinent les conditions optimales pour que les constructions et productions personnelles des élèves se développent au mieux et prennent la place qu' elles méritent.

1.Introduction and survey

When thinking about the title of the present lecture, words like constructivism and reconstructivism crossed my mind. I rejected them. I thought I would better avoid any allusion to last year's PME-XI discussion on constructivism and its interpretations (cf. PME-XI Proceedings),which I have neither the intention nor the talent to join.My aim is more concrete. The question I wish to tackle is: How to influence children to produce by themselves-albeit under guidance-their mathematical abstractions. (cp. Cobb, 1987) In order to answer it I will deal with successively:

-children's own production in mathematical instruction-what does it mean?(2);
-function of their own production in the teaching/learning process, with examples (3);
- own productions in education developmental search after reconstructible instruction (4);
A brief reflection on reconstructive learning will conclude the exposition (5).

2.What is own production?

In productive mathematics education children, guided by their teachers, construct and produce their own mathematics. The pupils' mathematical activity expresses itself in their construction and in the production resulting from reflexion on the constructions. Treffers (1987,p.260) has introduced this distinction, which according to himself is no matter of principle. Free production is rather the most pregnant way in which constructions express themselves.What, however, is own production? In order to answer this question we shall look out for the preconditions and circumstances under which productions emerge or may emerge in instruction.

By constructions we mean solving:

-relatively open problems which elicit- in Guilford's terms- divergent production, due to the great variety of solutions they admit, often at various levels of mathematisation; and

-incomplete problems, which before being solved require self-supplying of data or references.

An example of the first: How to divide two bars of chocolate among five children? An example of the second: A radio message on a 5 km queue at Bottleneck Bridge- how many cars may be involved?

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The construction space for own productions might even be wider:

-contriving own problems (easy, moderate, difficult) as a test paper or as a problem book about a theme or for a course, authored to serve the next cohort of pupils. An example, say, for grade one: Think out as many sums as you can with the result 5.

Finally there are border problems, that is, of constructive character but with a strong productive component,which require devising symbols, linguistic tools, notations, schemes, or models. In our illustrating problems stress is laid on the various functions to be fulfilled by own productions in the teaching/learning process (as well as in research). In fact, a production problem can involve more than one of these functions. The division according to functions is again a matter of stress rather than of principle.

3.Functions of own production in the teaching/learning process

3.1.Preliminary survey

If children's learning is to be expressed in their own production, its various functions have to be viewed under the aspect of instruction, that is, according to their didactical value (though of course from the learner's side). Without aspiring at completeness, we will distinguish the following functions:

-grasping the connection between phenomena in reality and the matching tool of description and organisation (horizontal mathematising) (3.2.);

-seizing the opportunities of continued organising and structuring of mathematical material (vertical mathematising) (3.3.);

-uncovering learning processes, and reversing wrong trends (3.4.);

-producing terminology, symbols, notations, schemes, and models serving both horizontal and vertical mathematisation (3.5.);

Each of these functions will be illustrated by examples and commented on. In all cases it will appear,that being productive in the mathematics lesson provokes both reflection and anticipation on the teaching/learning process.

The various functions will finally be considered within the broader context of course construction and education developmental research (4).

The whole will be concluded by a brief reflexion (5).

3.2 Grasping problems

Example :"The size of The Netherlands" (after Treffers, 1987), from the domain of calculation and meansuration by estimate:

Somebody affirms that the area of The Netherlands is 36,842 square meters, according to Larousse Encyclopedia, he says. What is your comment?

We will give an impression of the course of a lesson with 11 to 12 years olds who have received traditional (rules oriented) instruction, although in their last year (grade six) a few richer problems happened to emerge in the lessons.

The pupils start working while the teacher walks around, assists groups of pupils, and afterwards conducts the retrospective discussion.
At the start of the lesson the teacher had a brief talk with Mar:

Mar:Then I should first know what is a square meter.
Mar:I do know that a football-ground is a hectare.
T:How tall are you,Mar?
Mar:One meter seventy.

T:And now a square meter. Pay attention to "square".
Mar: I see. It is four times a meter (indicating a square).
T: This desk, is it as big as a square meter? (in fact it is 1.30 m . by 0.70).
Mar: No,it isn't square, so it is not a square meter..

Follows some explanation. Mar is progressing but time and again new obstacles arise; for instance, when The Netherlands is modelled into a rectangle of 200 km by 300 , and the area should be calculated. The estimated dimensions are to the point but 200x300 is done by column arithmetic. Mar's mathematical activities oscillate move between two extremes: intelligent estimating and thoughtless calculating.

Most of the pupils appear to know very well what size a square meter is, and understand decently what is area, but they still lack the mathematical attitude of trying a multiplication related to the 36,842 square meter, or starting at the other side, that is to make an estimate of the size of The Netherlands on the strength of available experience. They reproach the teacher walking around:"It is so big a number that one cannot imagine it, so there is nothing to comment."

They are given a hint; the size of a garden or so. It suffices to put them on the right track. In due course everybody is adjusted to explore wether the 36,842 square meters are possible. In retrospect the pupils deliver the comment (briefly summarised):

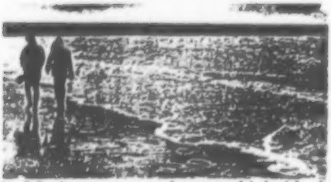
-If it were true, hundreds of people would live on a square meter because millions are living on the 36,842; so that is impossible;

-36,842 square meters is like a strip long 36 km and wide 1 m, and that is rather like a path through The Netherlands;

-36,842 square meters is not much more than a rectangle of 200 meter by 180 ,that is about six football-grouds-it is good for Gulliver's Lilliput;

-The Netherlands is about a rectangle of 200 km by 300 (cf. Mar's estimation), thus 36,842 square meters cannot be right.

All these comments are discussed. Almost all can follow the various reasonings and compare them with their own solutions. The last among these comments gives the teacher the opportunity to ask for the factual origin of the error. The whole group agrees that it should have been square kilometers. Then the teacher raises the question:How could right 36,842 square kilometers have come out?
In a final discussion objections are summarised:What about rivers, lakes, hills?Do they belong to the area of 36,842 square km?



What about the tides? Isn't our country larger at high tide than at low tide? How big can the difference be? Isn't the area to be considered as a variable?Finally the crucial question:" Wouldn't such a precise number be posssible on the strength of some model of The Netherlands?"
The teacher herself explains things: the fixed low tide line and the fixed map model define the calculation. In detail this model differs from reality. That then is how the size of The Netherlands is verified.

The foregoing was a good starting point. This is proved by a newspaper cutting:

> The classification obtained by this method has functioned for a certain time as shadow classification, never included into official tables. It is, however, attractive to have a closer look to this equivalising formula. Since it requires some arithmetic, let us restrict ourselves to The Netherlands. The country has about 14 million inhabitants, versus the 3 billions of the US, that is twohundred times as much. The area of The Netherlands is, say, 40,000 square meters, versus the 33,000 square kilometers of the US, that is thousand times as much. This weighed against each other yields for The Netherlands a population coefficient one fifth of that of the US.

In a test 312 future primary school teachers were asked a comment on this newspaper cutting. The scores were both revealing and distressing:

Correct : 18
Wrong :191
No answer:103

Many students performed their calculations exactly by means of column procedures. This indeed was the most essential shortcoming which could be observed,because this resulted in the production of failures which were not in the article. Other mistakes were sloppy arithmetic and the wrong processing of magnitudes and big numbers (Jacobs,1986).

Remarks:

Obviously the future teachers (as well as some of the pupils of the former example) lacked the notion that and how numerical data are anchored in reality and, with regard to measuring, did not have to their disposal reference points such as the size of a football-ground,the size of a country, the number of inhabitants. Mathematics education should aim at developing personal scales of familiar and lived through measures such as:

-the distance between home and school, also measured in time, walking and biking;
-one's own weight and stature;
-one's walking and biking distance per hour;
-the height of a house, a twenty stores building;
-the size of the playground, a football-ground, and so on.

Such personal scales, the richer the better, form reference frames for solving problems of the kind as presented.

What do the foregoing examples mean in the context of (re)constructive learning? Of course they have a merit of their own but in the present context they have been adduced because of their constructive and productive value.

Educated estimates and implicit experiential data made explicit, strengthen the grasp on problems, which is one of the functions of construction and production.Solving means tying connections between the real and the arithmetical world by means of mathematical modelling. Growing such connections helps developing a mathematical attitude, in particular horizontal mathematising, that is mathematisimng real world situations (cf.Treffers,1987).Almost all of the 312 future teachers lacked that mathematical attitude required to clean the mess of data in the newspaper cutting. This proves that the environment where they learned mathematics differed much from that of the school lesson.It is comparable with the direction in which the problem solving courses of Schoenfeld(1987,p.213) have been developed:

"
With hindsight, I realize that what I succeeded in doing in the most recent versions of my problem solving course was to create a microcosm of mathematical culture.Mathematics was the medium of exchange.We talked asbout mathematics, explained it to eachother, shared the false starts, enjoyed the interaction of personalities. In short, we became mathematical people."

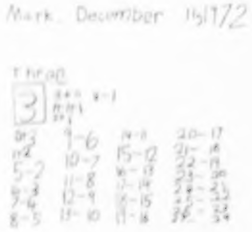
3.3.Seizing the opportunities of continued organising and structuring of mathematical material.

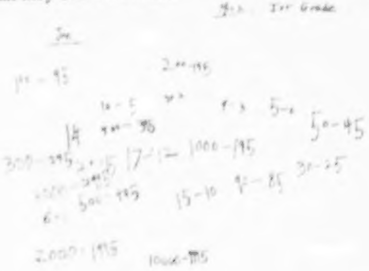
Pupil's own constructions and productrions is the mirror of the teaching/learning process, both for the teacher and the educational developer and researcher. Look to a few examples!

Example 1

Grossman(1975) reports about unexpected surprises caused by production tasks.She presents a few examples of work with first graders. We quote two of them with the teachers' comments(ibid.p.14-15).

"Mark was having trouble with arithmetic until I gave this assignment.He amazed me and he proved to himself that not only he could do arithmetic but that he couldn't stop doing it.(He handed in two extra papers on his own on subsequent days.) The other children loved the activity too.My feeling was one of constant amazement that they could do it all."


Mark December


1st Grade
50-45
17-12
1000-195

"I knew Jon was bright because he understood so well all that I taught in my structured lessons, wether I followed the syllabus or went just a little beyond it. However, I never suspected that he could handle number combinations in hundreds and thousands. There I was, teaching combinations up to twenty, limiting my expectations and the children's ceilings."

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Remarks:

The teachers' comments show that both boys had amply transgressed the limits of the scholastic domain. Mark's work still reveals traces showing how he reflected on his activities. After a hesitating start where he scanned the available arithmetic he screwed up courage, became selfconscious,wrote bigger, and sailed a fixed course through the system he built while constructing his problems. He transgressed the boundaries of the arithmetic lesson and produced his own structure. At home he continued intensively- the same Mark who had problems with arithmetic.

And then Jon! How much curtailed must he have been in his possibilities! He anticipated on sums, three grades higher in the curriculum; up to 10,000-9,995=5! Like Mark he worked systematically. Only his written report was a bit untidy.

Both of the boys reflected on what they had learned within the number system, and consequently they anticipated on the future of the teaching/learning process, the one farther than the other. The teachers were hold up the mirror of their instruction. Especially Mrs.S.(Jon's teacher) was conscious of this fact.

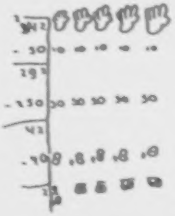
What would pupils' own constructions and production have mirrored in rich contexts of realistic instruction?The answer to this question can be found in numbers of publications (cf. Van den Brink, 1987).

Example 2:

A course of long division can be based on the principles of clever reckoning and estimating(cf Treffers, 1987). Let the start be

'342 stickers are fairly distributed among 5 children; how many does each of them get?'

In such a situation distributing shall be organised. First the stickers are handed out piecewise, but soon bigger shares are dispensed. The written report reflects the distributing pattern, which indicates the distribution process. Subsequent steps on the path of mathematising are predesigned.



In the second phase the children are soon satisfied with noting down one column only-'all get the same, indeed'. Other contexts are being introduced, among which that of grouping. After about 15 lessons the children work on different levels.

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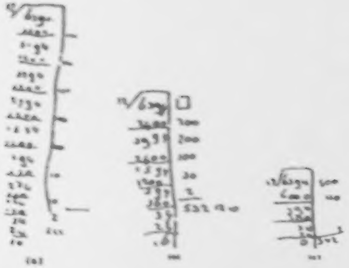

In the third phase the connection is made to decimals and fractions.Estimating according to powers of 1/10 becomes central but the procedure does not change essentially. Context dependant answers on divisions with remainder are not neglected. Again and again the opportunity is given to invent problems, among which illustrations of bare numerical divisions.An example:

6394:12, invent stories belonging to this sum such that the result is, respectively

532	532.84 rem.4
533	532.833333
532 rem. 10	about 530

At crucial points in the course it is asked to invent problems and to solve them by a slow longwinded manner as well as by a quick and short one - the pupils should learn te reflect on their learning process and to anticipate on even shorter procedures.

Remarks

With regard to contents the course, sketched above, of long division can be charcterised as follows:

-a process of clever calculating and estimating, integrated in context problems;

-a process of progressive mathematising arithmetical methods, in the present case by means of schematising and shortening.

Such an approach of division starts with the informal methods of the children, which are organised and structured. Construction and production play an important part in the process of progressive schematising and shortening, which are aspects of progressive mathematising. During the teaching/learning process the solution of applied problems are continuously subjected to inventarisation. Continuously the question of possible shortening is raised. The procedures arising in the course of shortening function in the course to be followed: beacons for those who nearly reached the same level of mathematising. The ultimate standard algorithm of long division is predesigned in this process as the utterly shortened procedure.

In a sense this mirrors the historical process of algorithmising long division (as well as the other operations on whole numbers; cf. Menninger, 1958). In fact the present course was at least practically inspired by the view on the historical development.

Comparative research undertaken in our country has proved that this approach is by far superior to the traditional one. An experimental group attained in half the time a result almost twice as good as a control group which had been taught traditionally (cf. Rengering,1983;Treffers,1987).

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	exp.	contr.
difficult divisions (zeros in dividend or divisor,etc.)	85%	45%
applications	70%	45%

Scores in long division - traditional method vs. progressive schematising.

The results on the traditional method have been confirmed by other research, also in other countries.

3.4 Uncovering learning processes and reversing wrong trends

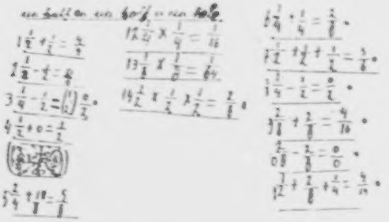
We have already noticed the diagnostic value of own constructions and productions, mirrors as it were, illustrating the teaching as well as the learning process.

At present we will consider the diagnostic value for the learning process, in particular cases where constructions and productions reveal wrong ideas and misconceptions.

Example

The class had elaborated and described two distribution situations (cp. Streefland ,1984;1987).It was quite early in the teaching/learning process,after a number of suchlike activities in the past. The teacher judged it the just moment to proceed to the first task of free production in this domain. The pupils were chalenged to think out such 'number sentences' as had been met in the distribution situations, that is, with halves, fourths and- for the courageous ones- eighths, with 'plus' and 'minus' ,maybe even with 'times', sums matching distributions.

Michael produced the following:



His work is typic for -world-wide- mistakes, I called it "N-distractors".(cf. Hart ,1981;Hasemann,1987; Streefland,1984; and many others).

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Remarks

The diagnosis is clear.The mistakes are the consequence of yielding to the temptation of whole numbers and their rules. It shows that the constitution of the mental object "fraction" has not progressed far enough to resist this temptation. Numerators and denominators were still operated upon separately; their conceptual interdependance was neglected.

The task had been set too early, at least for Michael. The concrete sources had been switched off prematurely. Stating this goes to the heart of the function here envisaged. In their own constructions and productions pupils can disclose their wrong ideas and misconceptions. In other words: Own constructions and productions unveil the -possibly wrong- personal theoretic basis of reflexion and anticipation in the teaching/learning process. This enhances the diagnostic value of the material. A correct diagnosis promises successful remediation both of learning and teaching.

As a matter of fact this is closely related to ideas in Sinclair's and Vergnaud's PME-XI addresses at Montreal. Indeed, what has happened? Michael reckoned among others 1/4+1/4=2/8 Remediation can start with eliciting a conflict. In the concrete (imaginable, meaningful) environment a new solution can be tried: Four children share two pizzas.Make a distribution-how much does each of them get? The solution may be ⊕ ⊕ (one fourth and one fourth are two fourths, which is one half).Some children don't experience this as a conflict.(cf. Streefland,1984; Hasemann, 1987). In these children's conception the result (still) depends on the solving method, that is, on the level at which the solution is conceived (concrete vs. symbolic).This level dependance is an example of what Sinclair (1987) named a " normative fact ", and Vergnaud (1987) " theory in action ".

3.5.Producing terminology, symbols, notations, schemes, and models serving both the horizontal and vertical mathematisation

Children learning mathematics can, by their constructions and productions, contribute to its working apparatus.

Example 1

Madell (1985) reported about the personal algorithms for subtraction, developed by pupils of the Village Community School in New York. Their 'natural' informal methods of performing the operation had the following characteristics:

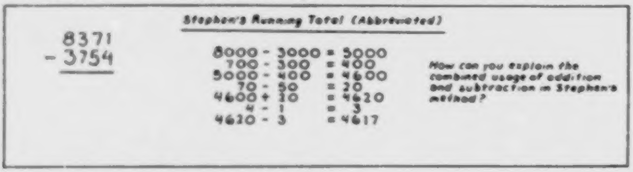
-both working (partly) per column and from left to right;

-working with position-values in stead of the numbers per position;

-working with deficits and borrowing from tens; nobody applied the standard procedure of borrowing;

-working along the lines of proceeding abbreviation.

Let us have a look at an example, reflecting the features just mentioned.



Stephen's Running Total (Abbreviated)

```
  8371
- 3754
```

```
8000 - 3000 = 5000
 700 - 300  = 400
5000 - 400  = 4600
  70 - 50   = 20
4600 + 20   = 4620
   4 - 1    =  3
4620 - 3    = 4617
```

How can you explain the combined usage of addition and subtraction in Stephen's method?

(quoted from Labinowicz,1987,p.381).

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This own invention of column subtraction may be used in the development of a standard procedure, deviating from the usual one.

8371
3754
5000-400+20-3=4617 (a)

The written report, however, could also have looked like this:

$$\begin{array}{ll} 8371 & \\ \underline{3754} & \\ 5000 & \\ -400 & 4600 \\ +20 & 4620 \\ \underline{-3} & 4617 \\ 4617 & \end{array}$$

(b)

This is positional working from the left to the right with 'deficits' which occur quite naturally.

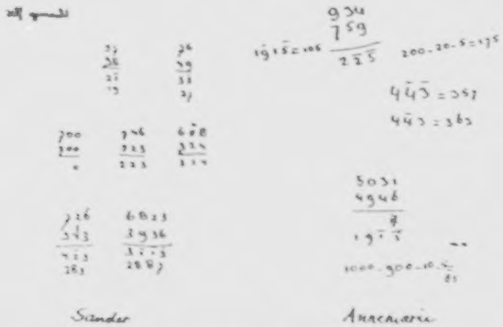
Method (b), however, can be shortened by means of a new notation, namely:

8371
3754
5④2③ (c)

Remarks

The self constructed notation served the development of an algorithm for column subtraction, while working from the left to the right, it is stated that 8,000 minus 3,000 equals 5,000,300 minus 700 is 400 short (notation ④ with the background reasoning that subtracting 300 is still possible , with the result 0, and subtracting 400 more brings ④ at the place; and so on.)

The deficits can also be indicated by parantheses or by upper dashes (cf. pupils work).


934
759
225
200-20-5=175
5031
4946
Sander
Annemarie

All notations are shortenings of the more extended ones in (a) or (b).

For the transition from 5423 to 4617 money can be used as a meaningful positional material: property 5,000 debt 400........This is a way one can use children's informal methods to start a process of algorithmising, based on clever calculating and steered by progressive mathematising. The production of new forms of notations mirrors the reflexion on the course of thought and creates the possibility to shorten the developed method.

Regularly such approaches are met with in publications.Apart from that, the possible consequencies for the outline of the programme for the algorithms usually are not recognized, dealt with and elaborated.

Anyhow Madell did not.According to what has been shown with respect to this, the usual algorithm for subtraction would have to leave the field.(after Treffers, Feijs en De Moor, 1988).

Example 2

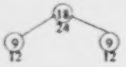
Dividing per unit and several units simultaneously in distribution situations is an oppotunity for children learning fractions to produce equivalences by themselves. In the distribution activity 1/4 and 1/4 go together with 1/2, and 1/2 can be decomposed (among others) into 1/4 and 1/4.

During our education developmental research (cf.4) pupils contrived such terms as 'hiding name' or 'conceal name' to indicate non-standard names for fractions.Such terms facilitated the communication but also described it efficiently (The Dutch word ' schuilnaam' sounds less 'learned' than English 'pseudonym').The quest for a fitting term for some (mathematical) phenomenon can elicit reflexion, as this example shows.

The most suitable propositions that were offered, also proved to have a long term predictive value.(cf.Treffers & Goffree ,1985;Streefland ,1988).

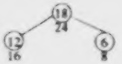
Example 3

In more extensive situations such as 'dividing 18 pizzas among 24 children' the actual distribution, wether pictorial or imagined, is too laborious.In our education developmental research some children got to use the service at tables as a means to reduce the situation to manageable proportions.Thinking about it they found out the symbol [illegible] for 24 children around a table with 18 pizzas.This made it possible to represent the service at tables on paper. It led to organising and structuring activities such as building schemes that expressed variations in table services. For instance:



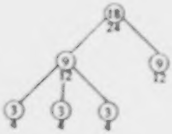
that is, two tables $\overset{\textcircled{9}}{12}$ instead of one $\overset{\textcircled{18}}{24}$

or



with the tables $\overset{\textcircled{12}}{16}$ and $\overset{\textcircled{6}}{8}$

This can be continued:



At all given moments this schemes-building can be interrupted. At one table everybody's fair share can be determined. In this way any distribution situation is being made accessible to the pupils via the double action of fair sharing at one table and the fair table service. Time and again the learner will reconstruct the food/consumers relation. This leads to an operational concept of fraction and ratio in their mutual relation. The scheme that is developed organises in an almost evident way the production of tables equivalent with regard to fairness. The food/consumers relation of the original situation shall be mentally continued at each step:ratio conservation. Schematising goes on with table service in the background. The context situation fulfils a model function: the model situation of table service becomes a situation model.(cf.Treffers &Goffree,1985;Streefland, 1986).

Remarks

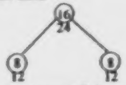
The quest for fitting symbols for distribution situations and schemes-building for table services supported by this symbol elicit reflexions uncovering the process of horizontal mathematisation.

Distribution situations are being located within mathematics. Anticipation is being encouraged by the opportunities of progressive schematisation, which emerge as naturally as in the example of long division. How does this happen?.

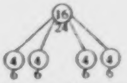
The self- contrived symbol and the patterns in which it occurs allow to compare situations with each other by decomposing them in equivalent partial tables which can more easily be compared, for instance tables with the same number of guests.The symbol is a metonym for the situation and the scheme is based on the situation model of table service, which functions as a cognitive process model.(cp. Greeno, 1976).

Continuously applying the scheme leads to two types of shortening, which uncover the reflexion on the own activities.

The first is scheme-conserving while the notation is simplified: equivalent branches, or at least the numbers are omitted so that the essentials of the table service are respected.The second is shortening in depth like



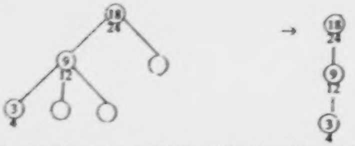
replaced by



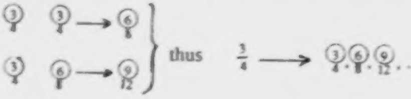
which changes the pattern of the table service with the numbers themselves and their common divisors steering the shortening.

This involves level-raising in the learning process. The provisionally highest level is attained when the pupils consciously and systematically start with the reduction by means of the greatest common divisor while knowing how to verbalise this idea.

Moreover this may lead the learner to focus on proportion tables



Starting from a given portion 3/4 and looking for matching table services connects 3/4 and $\frac{\textcircled{3}}{4}$ with each other; while both of them are still distinguished by the different notation. Pushing tables side by side generates new tables granting everybody the same portion



The next step indeed is proportion tables.(Streefland, 1985).

Example 4

19 years old Fynn (1976) told a fascinating story on six years old poor Anna. She had her own way to manage big numbers.She knew that big numbers could be made ever bigger but she lacked words to express them. When she would transgress the limits of millions and billions she , in order to continue, invented squillions.

Some fine day she told Fynn she could answer a squillion questions."Me too", Fynn said unimpressed, " but, among half of them wrong". "Not so," Anna said, " all will be good"." Idle nuts", Fynn thought, "nobody can and she the least ". She deserves rebuke. But Anna did not take it.

" How much is one plus one plus one ?"
" Three, of course ".
" How much is one plus two ?"
" Three."
" And eight minus five ?"
" Also three."
Fynn wondered what she was getting at.
" How much is eight minus six plus one ?"
" Three."
" How much is one hundred and three minus one hundred ?"

Fynn interrupted; he felt she was pulling his leg. She was inventing the problems on the spot and could go on that way untill the cows came home. Nevertheless Anna enthusiastically made her last move.

" How much is one half plus one half plus........"
Fynn had got the message.
" How many problems can be answered by three ?"
" Squillions," Fynn said.
" Isn't it funny, Fynn, every number is the answer to squillions of questions "
(hypothetical retranslation from the Dutch version)

Remarks

This own production reveals high level reflexion, typical for a mathematical attitude in the spirit of Krutetskii (1976) and Freudenthal (1978). The analysis is left to the reader. In the next section this example will be reconsidered.

The foregoing examples show the part played by the production of terminology, symbols, notations, schemes and models in the shaping of mathematisation, horizontal as well as vertical.

4 Education developmental research for the sake of reconstructible instruction

Up to now the stress has been on reconstructive learning, viewed in the learner's perspective. It started with informal notions and working methods. Reconstruction gradually moved the learner towards more formal mathematical notions, operations, structures.

(One of the examples was concerned with *unlearning*, wrong ideas and methods; notice that often too little attention is paid to the potential and need of unlearning in instruction (cf. Freudenthal, 1983).)

The import of reconstructive learning is also at the heart of Anna's message. Curriculum developers and researchers are seldom aware of such signals. Rather than seriously observing children and learning from their activities, their constructions and productions, they expect answers on questions and solutions of problems by prematurely theorizing within topical frameworks. Calls for change sounded time and again in the literatuire on development and research, are not listened to. The results of didactical research in teaching arithmetic are badly neglected. Fractions is a telling example: fresh starts with all old errors repeated. Nothing is learned from lessons such as taught by didacticions of mathematics like Freudenthal (1968)(1973), Hilton (1983) or Usiskin (1979).

A striking illustration of this fact is Brownell & Chazal (1932, p.24), who from the results of drill for the mastery of basic skills conclude: ".....the time and accuracy scores on Test B were better than on Test A, not because the month's drill had materially raised the level of the pupils' performance, not because drill had supplied more mature methods of thinking of the combinations, but because the old methods were employed with greater proficiency ".

By " old methods " the authors mean pupils' own informal solutions, which resist instruction against the grain.Wouldn't we have made greater progresses in our knowledge about childrens' mathematical learning if we had built on these telling results of research ?

Does reconstructive learning also apply longitudinally to class instruction ?Our reports related to the seize of The Netherlands, long division and table service provide indications for group learning processes.

In order to answer in the affirmative, we have to carry on education developmental research- research in action.

Such research aims at developing prototypes of courses and theory-building for teaching and learning in a certain subject area. Instruction experiments start with provisional material. The teaching/learning process is closely observed. Continual observation and registration of individual learning processes is at the heart of the research. What matters is that pupils' constructions and free productions are *used for building and shaping the teaching course*.

In the variety of children's possible proposals (look for the kind of problems to be used (2)) one gets a rich choice to find out what is the best fitting, the farthest prospective, and in the long run the most effective. Blocking and diverting material is eliminated.

This is no illusion. At ours as well as abroad courses have been developed in this way; for science, see Driver (1987) ;for mathematics, see Treffers (1987).

With the aid of children's constructions and productions a course for fractions closely tied to ratio and proportion has been developed (Streefland, 1988).

In this kind of design children, by their learning processes, decisively influence course development- this even extends to supposedly weak learners as some examples proved.

Once more: mathematics education is developed in een experimental situation, where pupils can contribute by their constructions and productions.

This nourishes the source for creating reconstructible instruction. The prototype can serve as a model for establishing and developing derived courses. Such potential instruction is predesigned in textbooks and manuals.Globally the used generative problems with pupils' usable long/term constructions and productions, which emerged in the developmental research, will mark the learning road for fresh pupils' cohorts. In particular the manual will prefigure the material to be expected from the pupils and help to reorganise it with the view on the sequel. This is a means to realise teachers aided reconstructible instruction.

Such teaching, rather than transfer of knowledge, is negotiation of meanings (Driver, 1987, p.8).No longer does the course represent the teaching contents but : ".....a programme of learning tasks, materials, and resources which enable students to reconstruct their models of the world to be closer to those of school " (mathematics; added by L.S.)(l.c. p.8).

5.Conclusion

The construction principle in education requires a significant part played by children's constructions. What does this mean for mathematics instruction?

a. With view on horizontal mathematising instruction maintains close bounds with *reality*. The basis in reality lends meaning to the mathematical notions, operations, and structures, envisaged to be produced; and it does so in a way that they become accessible to imagination and representation. Moreover the pupils get the opportunity to mathematise problem fields from reality by organising, visualising, schematising, structuring, shortening and so on.

Mathematics learnt shall be applicable in the reality.

b. Courses for fractions, ratio and proportion, for clever calculating and column arithmetic, and more general, courses that line up with each other should be strongly *interwoven* . Genuine reality can be organised mathematically in various ways. This connection should be respected.

c. In the learning process the children should acquire a manifold of *aids and tools* which help them to pass selfreliantly from the concrete to the formal (progressive mathematisation)- a supply of terminology, symbols, notations, schemes, and models. This distinguishes the intended approach from the structuralist one where vertical mathematising is overstressed and formal procedures are imposed (Dienes, 1973; Treffers, 1987).

d. On the road from the concrete to the formal, cooperation, negotiation, and discussion play an important part. Indeed the variety of constructions and productions brought about by the open approach invites negotiation of the various proposals for continued activity. For this reason instruction develops a full measure of *interactivity*.

e. Organising and structuring of the produced mathematical material with increasing efficiency is as much as possible the business of the pupils themselves.

Vertical mathematisation was well exemplified by long division and table service, in particular by the way how individual methods of clever calculations and notations were transformed into algorithms.

Such mathematics instruction has proved to be highly productive because it is supported by pupils' *own construction and production*.

Within realistic mathematics education a solid empiric basis is laid for the principle of constructivity by having the children contribute to course development. Horizontal and vertical mathematisation as observed in the historical learning process can be a source of inspiration. In the light of history reconstructive learning is realised on the individual as well as on the class level.

References:

Brink,J. van den, *Children as arithmetic book authors*, For the learning of mathematics, 7(2), 1987, 44-48.

Brownell W.A. & C.B. Chazal, *The effects of premature drill in third grade arithmetic*, Journal of Educational Research,vol.29,1935, 17-28.

Cobb,P., *Information-Processing Psychology and Mathematics Education*, Journal ofMathematical Behavior,6, 1987, 3-40.

Dienes, Z.P., *The six stages in the process of learning mathematics*, London, 1973.

Driver,R., *Changing Conceptions*, Paper presented at the International Seminar:Adolescent Development and School Science, London, 1987.

Freudenthal,H., *Why to teach mathematics so as to be useful?*, Educational Studies in Mathematics ,1(1,2), 1968, 38.

Freudenthal,H., *Mathematics as an Educational Task*, Dordrecht, 1973.

Freudenthal,H., *Weeding and Sowing. Preface to a Science of Mathematics Education*, Dordrecht, Boston, 1978.

Freudenthal,H., *Is heuristics a singular or a plural?*, R. Hershkowitz(ed), Proceedings of the Seventh International Conference PME, Rehovot,1983, 38-50.

Fynn, *Mister God,this is Anna*, London, 1976.

Greeno,J.G., *Cognitive Objectives of Instruction: Theory of knowledge for solving problems and answering questions*, D.Klahr(ed), Cognition and Instruction, Hillsdale(N.Y.), 1976,123-159.

Grossman,R., *Open-Ended Lessons Bring Unexpected Surprises*, Mathematics Teaching , 71, 1975, 14-15.

Hart,K.M., *Children's Understanding of Mathematics:11-16*, London, 1981.

Hasemann,K., *Pupils' Individual Concepts of Fractions and the Role of Conceptual Conflict in Conceptual Change*, E. Pehkonen(ed), Articles on Mathematics Education, Helsinki, 1987, 25-41.

Hilton,P., *Do we still need to teach fractions?*, Proceedings of the Fourth International Congress on Mathematical Education, Boston, 1983, 37-41.

Jacobs ,C:, *Rekenen op de PABO*, Utrecht, 1986.

Krutetskii,V.A., *The Psychology of Mathematical Abilities in Schoolchildren*, Chicago, 1976.

Labinowicz,E., *Learning from Children*, Menlo Park California a.o., 1985.

Madell, R., *Children's natural processes*, The Arithmetic Teacher, 32, 1985, 20-22.

Menninger,K., *Zahlwort und Ziffer*, Gottingen, 1958.

Rengering,J., *De Staartdeling.Een geintegreerde aanpak volgens het principe van progressieve schematisering*, Utrecht, 1983.

Schoenfeld,A.H.(ed), *Cognitive Science and Mathematics Education*, London, 1987.

Sinclair,H., *Constructivism and the psychology of mathematics*, Proceedings of PME-XI,Vol.1, Montreal, 1987, 28-42.

Streefland,L., *Unmasking N-distractors as a source of failures in learning fractions*, Proceedings of PME-8, Sydney, 1984, 142-152.

Streefland,L., *Search for the roots of ratio:Some thoughts on the long term learning process (Towards a ......theory ?) Part 2: The outline of the long term learning process*, Educational Studies in Mathematics,16, 1985, 75-94.

Streefland, L., *Rational Analysis of Realistic Mathematics Education as a Theoretical Source for Psychology: Fractions as a Paradigm*, European Journal of Psychology of Education, 1(2), 1986, 67-83.

Streefland,L., *Free production of fraction monographs*, Proceedings of PME-XI,Vol.1, Montreal, 1987, 405-410.

Streefland, L., *Realistisch Breukenonderwijs*, Utrecht, 1988.

Teule-Sensacq, P.& Vinrich,G., *Resolution de Problemes de Division au cycle Elementaire dans deux types de Situations Didactiques*, Educational Studies in Mathematics, 13 , 1982, 177-203.

Treffers,A.& Goffree,F., *Rational Analysis of Realistic Mathematics Education-The Wiskobas Program*, Proceedings of PME-9, Vol.2, Utrecht, 1985, 97-123.
Treffers,A.,*Three Dimensions. A model of goal and theory description in Mathematics Instruction-The Wiskobas Project*, Dordrecht, 1987 a.
Treffers,A.*Integrated Column Arithmetic According to Progressive Schematisation*, Educational Studies in Mathematics, 18, 1987 b, 125-145.
Treffers,A., *Beschrijving van eindtermen*, Toetsen,Eindtermen en opvattingen over Wiskunde-Onderwijs, Utrecht, 1987 c , 146-151.
Treffers,A., Feijs, E & De Moor, E.,*Proeve van een nationaal programma voor het reken-wiskundeonderwijs op de basisschool (5)*, Tijdschrift voor Nascholing en Onderzoek van het Rekenwiskundeonderwijs, 6(4), (in press).
Usiskin,Z.P.,*The Future of Fractions*, The Arithmetic Teacher ,1979, 18-20.
Vergnaud,G., *About Constructivism*, Proceedings of PME-XI, Vol.1, Montreal, 1987, 42-55.

RESEARCH REPORTS

PERCEPTIONS OF TEACHERS' QUESTIONING STYLES

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This paper outlines a small-scale research project which aims to explore the differing perceptions which teachers and pupils have of teachers' questioning styles. In particular, the research focuses on differing perceptions of the purposes of teachers' questions. The project is based on small group teaching with children in the primary age range (5-11). The first part of the paper gives a preliminary attempt to categorise the purposes of teachers' questions. The first stages of research, based on interviewing subjects about their reactions to video-taped material, are then described. In the final section, some early results are presented, indicating mis-matches between teachers' and pupils' perceptions, and ideas for the further development of the research are discussed.

CATEGORIES OF QUESTIONS

The considerable volume of research into classroom discourse indicates clearly that not only do teachers do most of the talking, but also that a large amount of teacher talk is made up of asking questions. (Hargie (1983) presents a review of such research.) Researchers in linguistics have categorised teachers' questions in a number of ways (Barnes (1969), Mishler (1972), Stubbs (1976)). These categorisations tend to focus on the linguistic form of the exchange, or on the type of question asked (e.g. open or closed), but do not take account of the purposes for which classroom questions are asked. These differ dramatically from the ways in which questions are used in everyday conversation. In particular, it is very common for teachers, and particularly teachers of mathematics, to ask questions to which they already know the answers. What is more, the pupils of whom the questions are asked know that the teacher already knows the answers.

In the context of normal conversation such behaviour would be considered very strange, and probably impolite. However, it may be seen as acceptable in other contexts where there is a perceived power relationship between the questioner and the person questioned (e.g. parent and child, drill sergeant and recruit. The most obvious purpose of such questioning is to find out whether the person questioned (afterwards referred to as 'the subject') knows the answer; the question is designed to test the subject's knowledge.

However, this is clearly not the only purpose for which teachers ask questions. The following table sets out a possible categorisation of the purposes of teachers' questions, viewed from an adult perspective.

category	characteristics	purpose
pseudo-questions	'....., do we?', '...., isn't it?' questions only requiring or allowing agreement	to establish acceptable behaviour, and social contact
genuine questions	questioner does not know the answer	to get information
testing questions	questioner does know the answer, and the subject knows this	to find out if the subject knows the answer
directing questions	questioner may or may not know the answer, the subject may or may not think that she does	to provoke the subject to think further about a problem

Three important sub-categories of directing questions are:

> structuring questions, typically a sequence of questions which 'activate' the subject's existing knowledge in such a way that new connections become clear.
>
> opening-up questions which suggest new areas of exploration, such as 'What would happen if ...', 'Why do you think ...?'
>
> checking questions which encourage pupils to think again about a statement, such as 'Are you sure?', 'Is that right?', 'Do you agree?'

There is a widely held, but largely unspoken, belief amongst teachers that questioning pupils is 'better' than straightforward exposition. This is discussed explicitly by Klinzing (1986), in a review of questioning research in Germany. If questioning is regarded as a single activity, without awareness of different styles and purposes of questions, there may be little foundation for this belief (Ainley (1986)). Because testing questions are so common, particularly in mathematics where answers are often seen as being clearly 'right' or 'wrong', there is a danger that pupils may perceive all teacher questions in this way. Such a perception would inevitably be detrimental to attempts to encourage discussion, investigative work or problem solving in mathematics: pupils will feel that the teacher always knows the 'right' answers to any question she asks, and furthermore that the teacher is always judging pupils by the answers they give. It is not surprising that pupils are reluctant to risk giving 'wrong' answers in these circumstances (Holt (1969)).

THE FIRST RESEARCH STUDY

The study was designed to try to reveal the perceptions which teachers and their pupils have of particular types of classroom questioning styles. For this purpose, video-taped extracts of teachers working with small groups of children were used. The extracts were chosen from the video tapes which accompany the Open University course, 'Developing Mathematical Thinking'. The extracts, each approximately five minutes long, were chosen to show differing styles of questioning on the part of the teachers, but all involved the introduction of new concepts by means of practical activities. Six different extracts were chosen involving children from age six to age eleven.

Four teachers were involved in the first stage of the research, all of whom were class teachers in primary schools. They were each shown three of the video-taped extracts, and interviewed about their general impressions of the teachers they saw, and about their perceptions of why particular questions were asked. Each teacher then allowed children to be withdrawn from their classes in groups of four (chosen by the teacher), to watch the video material. The children watched two of the extracts which their teacher had seen, and were interviewed about their reactions to them in the same way. All the interviews were recorded on audio tape.

Wherever possible, the children watched video material of pupils the same age as, or younger than, themselves, in the hope that this would make it relatively easy for them to understand the mathematical content of the extracts. Every attempt was made to reassure the children that they would not be questioned about the mathematics, to reduce the possibility of the questions posed by the researcher being perceived as 'teachers' questions'. In fact many of the children were eager to show how much of the mathematics they had understood, and it was usually clear from their responses when they were not able to follow the mathematical content of an extract. The children were also told that their teacher would not listen to the recordings of what they said. Several of the children were visibly relieved by this, and in fact many of them spontaneously made comments about their own teacher's use of questions, reactions to 'wrong' answers etc. Several other adults, both mathematics educators and parents of primary school children, watched some of the extracts and took part in interviews.

The interviews were conducted in an informal way, not following a set script. Each participant watched approximately half of each extract without interruption, and was then asked to comment on what they had seen. In the second part, the video tape was stopped at particular points and specific questions were asked. These questions required an opinion about why the teacher had asked a question, whether the teacher knew the answer before she asked a question, whether the teacher's reaction showed that a pupil had answered correctly or not, and so on. The questions were always posed in the form 'Why do you think ...?'. Three or four such questions were asked about each extract. The interviewer also tried to stimulate more general comments and discussion by asking questions such as 'Did you like that teacher?', 'How do you think those children are feeling?', and by inviting comparisons between the different teachers seen in the extracts.

An obvious difficulty with this research is that the interviewer had to engage in questioning participants directly. This inevitably raises difficulties because of the perceived purposes of the interviewer's questions, and the perceived roles of researcher and participant. The greatest danger is that the subject feels that they are being 'tested' by an interviewer who already 'knows the right answers'. This was not often apparent with the children who took part, though it is impossible to interpret the reasons why some children did not answer some of the questions. They may have felt anxious about giving a 'wrong' answer, or may have not understood the question, or may simply have been unable to express their opinions. Most of the children did not appear to be under any stress or anxiety during the interviews, and they talked willingly when they had some thing to say. But then, they are used to adults asking them questions all the time.

Some adult subjects showed much more unease about getting the answers 'right'. A mathematics educator who was familiar with the purpose of the research asked 'Am I giving you the right sort of answers?' towards the end of his interview. The following exchange indicates the perceptions of one (fairly senior) teacher, who watched the extracts with a colleague.

Researcher: (After stopping video tape) *Why do you think the teacher asked that question?*
Teacher 1: *Well surely it's because* ... (comments about content of the lesson)
Researcher: (No response - hoping for comments from teacher 2)
Teacher 1: *Oh well, obviously it wasn't!*

INITIAL RESULTS AND FUTURE PLANS

The initial plan in analysing the data gathered from the interviews was to try to identify how different subjects perceived the teachers' questions, in terms of the categories outlined above. This was to be done in two ways: comparing the responses of adults generally with those of children, and comparing the responses of particular teachers with those of children in their classes. The first stage was partly a feasibility study for more extensive research.

Three important problems arose. First, since none of the participants were shown the categories before the interviews, their responses do not refer directly to them. It is therefore not easy to interpret many of the responses in terms of the categories. It would be possible to rephrase some of the specific questions used in the interviews so that they refer explicitly to the given categories, or so that they become multiple choice questions. This would, however, limit the range of possible responses, and would also mean that different formats might have to be used for adult and child subjects.

Secondly, it is often only possible to consider the purpose of a particular question within the context of a longer exchange. On several occasions adult subjects offered two or more responses to the question 'why do you think the teacher asked that?', dependent on how the conversation might develop.

Thirdly, both children and adults tended to identify strongly with the people they saw in the extracts, and as a result their responses tended to be related to the mathematical content or to personalities. Many of the children became too involved with solving the mathematical problems to give any attention to particular things that were said. Typical responses from adults were,

- ... *she's really getting on my nerves,*
- ... *well, I wouldn't have started like that,*
- ... *she should have used a bigger shape; the children can't see it!*

Despite these difficulties the study has demonstrated the feasibility of the method for more extensive research and some fairly clear results emerged. One initial conjecture was that children would tend to interpret many teacher questions as testing questions,and in particular that both directing questions and genuine questions might be perceived in this way. The nature of the extracts meant that there were few occasions on which teachers asked what appeared to be genuine questions, and only one of these was highlighted by a specific question from the researcher. All of the children interviewed thought that the teacher did not already know the answer to this question.

However there were several instances where teachers interpreted a question as a directing question, while some of the children saw the same question as a testing question. There is insufficient space to give a detailed context for each incident, but the following examples give an indication of the differing responses.

Teacher question in extract: *How many does that make?*
Interviewer's question: *Why do you think she asked that?*
Teacher response: *She wants them to check ... so that they're working it out for themselves ... She's imposing a systematic approach.*
Child response: *To see if they knew.*

Teacher question in extract: *Seventeen what?*
Interviewer's question: *Why do you think he asked that?*
Teacher response: *Because they could have got lost ... he's wanting them to check.*
Child response: *To see if they knew ... if it was right.*

Teacher question in extract: *What's the next one? ... and the next?... and the next one?*

Interviewer's question: *The teacher asked the same question several times; why do you think she did that?*
Teacher response: *...she was doing it to build up the idea of a pattern. She was hoping they would get used to the sequence.*
Child response: *To make sure they'd got it right.*
To let her know ... what they'd learnt.

On other occasions, children were strikingly acute in recognising questions which seemed to be directing/checking questions, even when these were disguised.

Teacher question in extract: *Six? (Repeating a child's answer)*
Interviewer's question: *Do you think the teacher thought she gave the right answer?*
Teacher response: *He was making her think again ... a way of making her check that she'd said what she really meant to say.*
Child response: *It sounded as though he said 'don't be stupid'.*

For all of the examples quoted above, there were other examples where children's responses indicated that their perceptions of the purposes of questions were very similar to the perceptions of teachers and other adults.

A much clearer difference between the perceptions of children and those of adults, and one which was totally unexpected at the start of the research, concerned the pace of questioning in two of the extracts. This emerged from general comments about teachers, which were invited both halfway through each extract, and at the end. No comments were recorded from adults which contradicted the overall tone of those given below, although the pace of questioning was not discussed in every interview. One group of children did show a different response to extract D, which is discussed later.

Extract C

Teacher/adult comments: *He certainly gives them plenty of thinking time, ... it seemed painfully slow ...*
Child comments: *He doesn't give them enough chance.*
He doesn't give them enough time to work it out ... he rushes them all the time ... he wants to get on to something else.

Extract D

Teacher/adult comments: *...there was very little time to reply ...*
I don't know why she kept asking questions ...
... they weren't allowed much thinking time.
Child comments: *She didn't, like, rush you into it ... she asked you questions.*
... if you didn't understand it, she talked it through.
She gave you time ...she never said 'what is it?' straightaway.

The differing perceptions reflected in these comments is particularly striking because it is reversed in the two incidents. A possible explanation for this lies in the age and experience of the children interviewed. The children who commented on extract C were clearly having difficulty following the mathematics involved (subtraction of hundreds, tens and units). The children whose comments on extract D are recorded were eleven-year-olds, and found the investigation in the extract relatively easy. One group of less able children made comments on the same extract which indicated that they felt the teacher was hurrying the children. Further research is planned using the same extracts to explore this conjecture.

The continuation of this research is based on making video recordings of several teachers working with children from their own classes. The activity used will be the same in each case, so that comparison between situations will be simplified. The teachers and children who take part will be shown their own tape, and possibly also tape of other groups, and a similar interview technique will be used to compare perceptions of particular conversations.

References

Ainley J. (1986)	'Telling Questions', *Mathematics Teaching*, 118, pp. 24-27.
Barnes D. et al. (1969)	*Language, the Learner and the School*, Penguin, Harmondsworth.
Hargie O.D.W. (1983)	'The importance of teacher questions in the clasroom', in Stubbs and Hillier (eds), *Readings on Language, Schools and Classrooms*. Methuen, London.
Holt J. (1969)	*How Children Fail*, Penguin, Harmondsworth.
Klinzing H.G. (1986) & Klinzing-Eurich G.	'Teacher Questioning', *Questioning Exchange*, Vol 1 (3), pp. 199-214.
Mishler E. (1972)	'Implications of teacher-strategies for language and cognition', in Cazden et al. (eds), *Functions of Language in the Classroom*, Teachers College Press, New York.
Stubbs M. (1976)	'Keeping in touch: some functions of teacher talk', in Stubbs and Delamont (eds), *Explorations in Classroom Observation*, Wiley, London.

TEACHER CHANGE AS A RESULT OF COUNSELLING

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In a project based on in-school in-service teacher education, the project staff defined some criteria for effective teaching of mathematics--for example, variety in teaching style, cognitive level of questioning, etc.
The analysis of project records led to the creation of teacher profiles (ineffective, effective and desirable) which were used to evaluate the effect of the project on each teacher.
As a result of two years of project activities we can report progress in both teacher change and student achievement.

INTRODUCTION

This paper describes the process of teacher change as a result of an ongoing three-year project aimed at improving mathematics teaching and student achievement in two Israeli urban schools. The target population of this project is about 20 teachers and their 7th through 9th grade students, who were studying mathematics in about 80 different classes at the upper two (of three) ability levels. Four counsellors paid weekly visits to the two participating schools, and were employed part-time for the project by the Department of Science Teaching at the Weizmann Institute. For the rest of the time, they were practising teachers in schools of their own.

The project's main intervention relied on observation-based counselling. Each of the four counsellors worked with four to six teachers. As in other counselling projects (Apelman, 1981) we repeatedly emphasised to all involved, that the counsellors did not have, and were not interested in any administrative supervising authority, and that their relationship with the teachers was strictly professional.

PROJECT DESCRIPTION

During the first two years of the project, a teacher's lesson was observed at intervals of two or three weeks. All visits were scheduled in advance and dependent on teacher consent. Notes were taken freely during the observed lesson, and short, one-sheet observation forms were completed the same day. Each observation was followed by a ten- to forty-minute discussion between counsellor and teacher of the observed lesson.

TEACHER CHANGE AS A RESULT OF COUNSELLING

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In a project based on in-school in-service teacher education, the project staff defined some criteria for effective teaching of mathematics--for example, variety in teaching style, cognitive level of questioning, etc.
The analysis of project records led to the creation of teacher profiles (ineffective, effective and desirable) which were used to evaluate the effect of the project on each teacher.
As a result of two years of project activities we can report progress in both teacher change and student achievement.

INTRODUCTION

This paper describes the process of teacher change as a result of an ongoing three-year project aimed at improving mathematics teaching and student achievement in two Israeli urban schools. The target population of this project is about 20 teachers and their 7th through 9th grade students, who were studying mathematics in about 80 different classes at the upper two (of three) ability levels. Four counsellors paid weekly visits to the two participating schools, and were employed part-time for the project by the Department of Science Teaching at the Weizmann Institute. For the rest of the time, they were practising teachers in schools of their own.

The project's main intervention relied on observation-based counselling. Each of the four counsellors worked with four to six teachers. As in other counselling projects (Apelman, 1981) we repeatedly emphasised to all involved, that the counsellors did not have, and were not interested in any administrative supervising authority, and that their relationship with the teachers was strictly professional.

PROJECT DESCRIPTION

During the first two years of the project, a teacher's lesson was observed at intervals of two or three weeks. All visits were scheduled in advance and dependent on teacher consent. Notes were taken freely during the observed lesson, and short, one-sheet observation forms were completed the same day. Each observation was followed by a ten- to forty-minute discussion between counsellor and teacher of the observed lesson.

In addition to observation-based counselling, the project staff conducted the following activities:

--- Weekly workshops on mathematical and didactical topics of general interest (e.g., the treatment of lack of prerequisite knowledge, design of class tests to obtain cognitively balanced set of test items, checking homework, using calculators, curriculum planning, techniques for teaching selected topics, etc.).

---"Open lessons" - i.e., lessons taught by one of the teachers and observed and discussed by the rest of the mathematics teaching staff.

The analysis of written records and personal accounts led to the definition of the following flaws in teaching style and lesson content that tended to turn up repeatedly in the observed classrooms:

Teaching style.

- Lack of variety -- an excessive use of frontal teaching or ineffective seatwork during most of the class period.
- Lack of teaching aids -- their complete absence or ineffective implementation.
- Inefficient homework checking -- mainly the allotment of excessive time and the use of ineffective strategies.

Lesson content.

- Lack of objectives -- sloppy or non-existent planning of lessons.
- Lack of variety in questioning -- the dedication of whole lessons to questions on a uniformly low or too high cognitive level.
- Ignoring lack of prerequisite knowledge -- improper or complete lack of treatment of student deficiencies in this field.
- Lack of teacher knowledge of mathematical content.

Characteristically, the teacher-counsellor sessions that followed observed lessons concentrated less on discussing "what went wrong?" but rather on working out things that "might be done differently".

PROCESS OF TEACHER CHANGE

An analysis of the records of classroom observation and subsequent counselling led to the following categorization of teaching as (1) effective, (2) ineffective, or (3) desirable. Each of these categories will be described, and change of teaching (i.e., movement within or between categories) as a result of the project intervention will be considered and illustrated by examples.

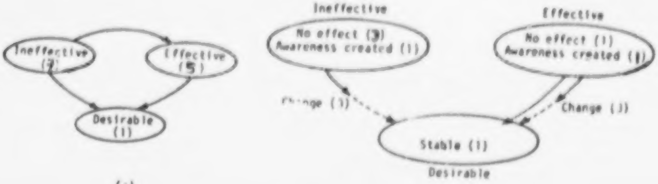
1. Effective teaching exhibits many of the following qualities: good class management, clear and correct mathematical content, well-planned lessons resulting in a feeling of learning, good learning environment, continual student appraisal with adaptation of teaching accordingly, and well designed homework.
2. Ineffective teaching will, naturally, contain a high percentage of the opposite "qualities".
3. The teaching considered by the project staff as desirable would have, in addition to the characteristics of effective teaching, some of the following qualities: varied teaching style, high student involvement, effective use of teaching aids, inclusion of high cognitive level questions.

The purpose of the project was to cause change from effective, or ineffective teaching towards a "desirable" style. Diagram 1a presents schematically the three categories of teaching described above and the potential directions of change. The number of teachers in each category, as observed during the first half of the year, are indicated in parentheses.

The process of change undoubtedly requires a long period of time and considerable effort (see also Ryan, 1984). As indicated by Blanchard (1981), before any actual change in teaching occurs, a teacher's awareness of a short-fall in his/her teaching must exist or be created. This awareness can be developed and observed in counselling sessions or informal counsellor-teacher conversation, but does not necessarily manifest itself in classroom teaching.

Observation records also indicate that changes in teaching are not monotonic. In most cases, the transition from one category of teaching to another is not sudden and definitive. Lapses into a previous stage were observed in spite of evidence that the overall direction was positive. It would seem clear that a period of time is needed to stabilize any progress made. Change should, therefore, be considered a process of gradual increase in desirable characteristics of teaching, and a corresponding decrease in practices characteristic of ineffective teaching. It is also clear that the time required to develop awareness of teaching "deficiencies", to create change and to achieve stability (not in the sense of stagnation, but in the sense of overall desirable teaching behavior, without serious lapses) is a variable dependent on each teacher.

Diagram 1b presents an overview of change (or lack of change) as observed during the project's first two years, with the corresponding numbers of the participating teachers in parentheses.



(a)

(b)

Diagram 1: Categories of teachers by observed change as a result of project intervention during two years.
(The numbers refer to teachers who participated in the project during both years.)

The following three case studies illustrate the categories of teaching mentioned above, and the effect of counselling on the corresponding teachers.

Teacher A entered the project as a very good teacher using varied teaching strategies and relating well to each pupil individually. (That is, the qualities of her teaching were considered "desirable".) The counsellor provided her with more varied ideas and, most important, gave her positive feedback and encouragement to continue on her way. This encouragement enabled A to reach her potential, and play an active role in the process of changing the other staff members.

In contrast, teachers B and C started as ineffective teachers. They would open the book, and assign the next page, without deciding which exercises are necessary or desirable. Consequently, most of their lessons were devoted to checking homework. Their explanations were clear but technical, and were much like cookbook recipes -- "this is how you do it, and now to the same." A typical lesson would comprise an explanation followed by a disorganised "ping-pong" of questions and answers between these teachers and the students. Students in their classes tended to be bored and disruptive.

By the end of the first year, B willingly accepted his counsellor's comments, but no actual change in his teaching style or lesson contents had been observed. B's awareness of short-falls in his teaching had been created, but this had not led to change. During the second year, B's teaching in his 7th grade classes showed a marked improvement. He had internalized the teaching suggestions and was utilizing them effectively, even adding his own ideas. In the first year of the project, he was teaching only 7th grade classes and so only those subjects were discussed. Teacher B did not make the transfer to his 8th grade classes and during the second year his teaching and his students' reaction in those classes continued to be problematic.

This year--the third year--Teacher B extended his change in teaching to his 8th grade classes as well and now is quite an effective teacher. He has learned to plan his lessons, make use of teaching aids and pose questions at varied cognitive levels.

To conclude, Teacher B took almost a year each time to internalize the advice given, and was unable to transfer ideas from grade to grade. Initially, he was unsympathetic to the counsellor's workshops, but now is one of her biggest supporters. He still needs to stabilize his new pattern of teaching.

Teacher C, however, is still stagnating at the end of two years. He sees no reason to change his teaching, and considers his style appropriate for students of junior high school age. We have not yet found a way of arousing his awareness.

The following three tables provide data that illustrate changes in teaching style, in cognitive level of questioning and in the use of teaching aids for the above three teachers.

Table 1: Change in teaching styles measured in terms of time spent in each lesson.

	FRONTAL TEACHING		INDIVIDUAL WORK	
	start of project	end of 2nd year	start of project	end of 2nd year
A	half	half	half	half
B	almost all	half	none	half
C	almost all	almost all	none	a small amount

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Table 2: Change in cognitive level of questioning measured as part of overall questioning in each lesson.

	TECHNICAL QUENSTIONS		QUESTIONS REQUIRING COMPREHENSION	
	start of project	end of 2nd year	start of project	end of 2nd year
A	more then half	less than half	some	over half
B	almost all	half	very few	half
C	almost all	half	none	half

Table 3: Change in use of teaching aids

	start of project	end of 2nd year
A	blkbrd, colored chalk, worksheets, games, number lines	blkbrd, worksheets, flashcards, calculators, games
B	blkbrd, colored chalk, worksheets, games	blkbrd, poster, cut-out geometric shapes, workcards, worksheets
C	blkbrd	blkbrd, colored chalk

TEACHER CHANGE AND STUDENT ACHIEVEMENT

The ultimate goal of this project was to improve student achievement. We believe that this can be done only by working directly with teachers on improving their style of teaching. The results presented in Table 4 support this belief. The results are based on two achievement tests - administered to the seventh graders of one of the two participating schools at the beginning and the end of the school year. As can be seen, teacher change had a considerable impact on student achievement. The classes of the "unchanged" teachers started the year at the same or even at higher level as compared to the others, but achieved less by the end of the year.

Table 4: Influence of teacher change on student achievement (in percentages)

categorization of teaching	1st test common fractions (at the start of 7th grade)	2nd test 7th grade mathematics (at the end of 7th grade)
desirable	58	68
change-ineffective	60	60
change-effective	57	63
no change-inffective	59	55
no change-effective	68	58

CONCLUSION

The model for in-school in-service teaching counselling described above was designed as a means of improving student achievement by increasing the effectiveness and the level of teaching mathematics. The counsellors isolated a relatively small number of serious flaws in teaching style that tended to turn up repeatedly in the observed classrooms, and through rather intensive intervention, attempted to induce change in a direction considered by them "desirable".

As a result of two years of project activities we can report progress in both teacher change and student achievement. However, the progress made is only partial and any change obtained needs further stabilization.

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REFERENCES

Apelman, M. (1981): The role of the advisor in the inservice education of elementary teachers: A case study. - Unpublished doctoral disertation. University of Colorado.

Blanchard, K. H.; Zigartmii, P. (1981): Models for change in schools. - In: Changing school mathematics: A responsive process/Price, J.; Gawronski, J.D. (Eds.). Reston, Virginia: The National Council of Teachers of Mathematics, (pp. 36-51).

Ryan, D. W. (1984): Improving the content and process of pre- and in-service teacher training. - In: Evaluation in Education v. 8 (2), p. 113-131.

CODIDACTIC SYSTEM IN THE COURSE OF MATHEMATICS: HOW TO INTRODUCE IT ?

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In the preceding research report, at PME 11, we presented a teaching experimentation in the course of mathematics, at University level [2]: a large place for uncertainty is left within the teaching process, institutionalized by the notion of conjecture, the validation of which, and sometimes even the production, is devoluted to the collectivity of students (these conjectures concern parts of the mathematical knowledge that students must learn during the year in their curriculum), proof arguments given by a student aren't adressed to the teacher, but, in some form of "scientific debate", to the other students. To introduce this alteration of the usual didactic "coutume" in the classroom, we use some definite actions as regards the role and rules of debate and the constitution of some scientific autonomy for each student, that we analyze in this report.

§1 "Problématique" and general theoretical framework.

Our general "problematique" in this experimentation has been detailed in the preceeding report, and we recall it briefly here: In mathematical productions of many students in the beginning of their first year at University it is frequently observed that the control of meaning does not seem primary. Often the syntactic characters prevail over the semantic ones.

Another observation is the lack of interest for proof as a functional tool: it is only a formal exercise to be done for the teacher. It seems to have no deep necessity.

These observations, for students that begin scientific studies at University, show that mathematics aren't acknowledged as a scientific subject which has been developped for playing a role in the resolution of problems and the understanding of reality.

We will now define our theoretical framework:

-First the constructivist model as model of knowledge acquisition: students construct their own knowledge through desequilibration of an old one in problems, interactions, conflicts, and reequilibration in which mathematic knowledge, other students, teacher are involved: we consider the role of the group of students in the learning processes to be very important, and especially in the construction of meaning.

- Metamathematics factors, such as representation systems of what are mathematics, how one learns mathematics, are very important in learning processes, especially when a student is solving problems. Moreover we think that, at University level at least, it is possible to act on these factors by an explicit work [5].

- Lastly , we think that a true learning of mathematics, in its scientific extension, must include the constitution of a "learner's epistemology" which is not only a school one: by "learner's epistemology", we mean the set of problems, situations, that , according to the personal experience of a student, has come with the introduction , the progressive constitution of a concept, and therefore gives, for this particular student a particular meaning to this concept.We can do the same remark as regards the image of mathematics in general given by usual teaching practices: Formulation of conjectures and proposition of proofs are two fundamental aspects of the professional mathematician's work. These practices, which constitute real mathematics, are generally absent in teaching process: mathematics are presented as an achieved body, where "all is certitude ". The epistemology generated by such teaching practices is diametrically opposed with mathematical reality.

So we think that the necessity, the functionality of proof can only appear in a situation in which the students meet uncertainty about the truth of really important and useful (for them) mathematical propositions, not only about more or less anecdotal exercises: Scientific Debate takes place during the lesson about these statements.

In this report we will study a more particular problem, inside the whole experimentation: students, at their entry at University have some customs, as regards mathematics, learning of mathematics, or proof in mathematics. They have something like a model of the respective roles of teacher and pupils in a classroom, what behaviour can be expected of eachone in this little "society" ... in short we call it the "coutume". Is it possible to change this "coutume" and try to constitute anotherone, and how can we manage such an evolution in few weeks at the beginning of the academic year in order to use a new "coutume", which is characterized as the codidactic one [2] [3], during the greatest part ?

S2 Description of the experimentation:

The experimentation of Grenoble takes place in a section of about one hundred students in the so-called Deug A (students are in their first year at University, they have courses in mathematics, informatics, physics, chemistry) The experimentation goes on during the whole academic year: lessons are given to the whole section in an ordinary amphitheatre. This section is one of four sections of DeugA1: the others have different organisation. The students are not selected: they receive a short information about the different teaching methods before they entry in a section, and they make their choice. This experimentation began in 1984.

(1) First we give some information about the presentation of the sections from which students chose in which one they'll register. This year, and more or less in the same way the previous, it was done in two times: First in July every student that wanted to go at University next October in a section of Deug A1 recieved a paper in which the four sections presented themselves. In our presentation, we emphazised that

"some fundamental concepts in mathematics are built in interaction with students from problems often linked with modelisation of physical problems", "a qualitative analysis of concepts is developped through discussions and debates managed by the teacher",

and we proposed "... the following contract: teachers favour different kind of expression of students ... students, in turn, accept to imply themselves in the knowledge they have to learn, and to go into an interactive practice in which error is not considered a fault, nor its analysis a loose of time".

Next in October , before the final registration, there was a presentation of each section by the teachers : it consisted in a correction of some exercises students had to solve during summer holidays (in math, physics, chemistry). During such presentation students can really see, in action, the different methods used in different sections. In our presentation (about half an hour long), the teacher chose to correct only a little part of the exercises, a conjecture:

" limits of $x \sin(1/x)$ and $x \cos(1/x)$, in 0, are equal "

and began to organize a debate about the validity of this statement: "who thinks it's true? who thinks it's false? who can't give an answer? ". During this short debate, some wrong answers were considered, discussed, and students encouraged to give their advice, and their arguments.

(2)The first courses of mathematics: during the first lesson of the year, the teacher presented the new "coutume" that was going to be used during the course. This new coutume was partly explicit, and partly used without explanation, about a very simple mathematical problem. The teacher asked a question , then organized a debate, without explicit rules, about the validity of answers, and concluded more or less with students. Last two years the question was:

"let $f:E \quad F$ be an application, X, Y parts of E, is there any relation between $f(X)$, $f(Y)$, and $f(X \cup Y)$? I give you 2 or 3 minutes of reflexion to propose some conjecture... "

Typically, 15 or more statements were proposed by students, many of them false, or not clear, using "+" instead of "U" for instance. The teacher write these statements on the blackboard without any comment, trying to give by his attitude no advice about their validity (this is a first rule, see [2]) and immediately some students wanted to refute some propositions: the teacher instituted a rule: all statements are written before the debate began ...

Every year, there are two or three such lessons, along which some rules are progressively put, very simple ones: speak loudly, speak for the other students, listen to one another ... or more hidden others: the teacher, since the first lesson, has some precise attitudes, for instance:

- facing a question, if he thinks that it's a question for many students, he asks for the production of a conjecture;

-when a problem has been put, he takes answers as conjectures proposed by students, write them on the blackboard without any comment. Then he waits some time (2 or 3 minutes, 5 if necessary) for reflection and asks for a vote:" Who thinks this statement true, false, who can't decide, or refuses to decide ?", then for mathematical arguments for, or against, the statement.

In such a way, the idea that to formulate a conjecture is authorized, and useful, the idea that a mistake is not a fault, but a normal stage in learning and doing science ... are settled in the community of students.

It is observed , at this stage, that very frequently the teacher has to close a debate by giving his opinion about the question, because students can't decide and convince one another.

(3) It is time to have a special lesson that we call "the circuit": the aim of this lesson is to give students means to refute a statement, and as a consequence some scientific autonomy within their community. This lesson produces a rule:

"In mathematics a statement is true if and only if it has no counter-example".

Observation shows that this rule is not easily accepted by all students: if a statement is true in all cases except one, it is unusual, in life time, to tell that it's false! "The circuit" uses a scheme of a very simple electric circuit to formulate conjectures, refute them, and progressively reach the logical rules used in mathematics: it is very important that the math context do not hide the logical problems.

(4) Generally at most four lessons (of 2 hours) have been spent at this stage, and the main rules of the new "coutume" have been used and some explicitly stated to students. The teacher have then to reinforce them by a frequent use (there is not a debate in every lesson), and some weeks later to recall some of them if necessary, often having some examples of very fruitful debates to support the interest of this form of mathematics teaching.

§3 Methodology of the study:

To analyze this part of the experimentation, we use classical methods [2]:

(1) An open questionnaire: at the end of the presentation of the section (see §2), students were asked to answer a questionnaire and to give us back before the first mathematical course of the section . It is a long questionnaire (13 questions) about mathematics, teaching mathematics, learning mathematics. As regards the problem presented here, we use it mainly to study

-what are their ideas about "good teacher", "good teaching method", "how to be a good student in maths", "whether collective work is useful to learn mathematics" in order to have an insight about the "coutume" they have before their entry in the section. [4]

-why they chose our section, what change in teaching they expected at their arrival at University, in order to know what variation of the "coutume" they already have anticipated (some answers about this second question can appear in the first one)

-what are their ideas about "doing mathematics".

(2) The lectures are recorded, observed and analyzed through a definite grid of interpretation, to see what is creating obstacles or facilitating the installation of the new "coutume", how the behaviour of students is transformed along successive lessons, how many of them really participate to debates.

More precisely we try to answer the following questions: what rules are introduced by the teacher, at what stage of the course, through what action, are they implicite or rather explicitely stated ? What kind of questions do students ask? Are they of the old "coutume" (i.e. adressed to the teacher, or only to ask for a detail, for an immediate answer), are they of a less usual kind (i.e. conjectures proposed to the group of students, not only of school form, about epistemology of concepts ...) ? What kind of answers do students give ? What is the global attitude of the group, passively listening the course, actively trying to find conjectures, or arguing about conjectures, or solving problems ... ?

§4 Some results and perspectives.

First we give some indications from the questionnaire in order to get a kind of "state of the coutume":

- A good mathematic teacher is first a "good teacher": interesting, convincing, clear, quiet (52%). In 11% of answers only we find such things like "helps students to reflect, to participate..."

- A good mathematic student must, above all, "work", "train himself" (85%), "know his math course" (45%). In contrast with the 11% just quoted, 45% of students think a good student must have an active attitude as regards knowledge: "to have some critical sense, to search, to argue, to deepen ..."

As a reference, trying to know whether our students are self-selected on account of the characters of the section, we use an analogous study, at the pre-universitary level [4]: a great majority of pupils (75%) uses expressions like "clear, giving explanations at pupil's level..." in order to describe a good math teacher, and " to work" is the first quality of a good math learner (76%) , 34% only uses expressions of the kind of "to work with an active attitude".

We see that there is only a little difference, if there is one, between students enterring our section and these reference pupils: if there is a self-selection, it does not concern these characters (probably there is only a little potential in this domain).

On the other hand, we notice that these students are open to change their working methods: 93% think that collective resolution of problems may be fruitful, and 64% that collective work is potentially interesting for general learning of math. Their expectations, at the beginning of Universitary studies are less definite: more responsability, liberty, less contact with teachers. Less than 20% expect really a more active participation within math course. Lastly, the epistemology underlying the "coutume" is a little problematic one: less than one third of students uses expressions containing the word "problem" as regards the activity of professional mathematicians, though in the same time almost everyone thinks math are useful (in physics, chemistry, economy, current life...).

As regards the installation of the new "coutume" we have observed that after the "circuit" lesson, a great part of students were really active in debates and the main rules were installed and devoluted to the collectivity: it was frequently students that recall it if necessary. So we think that the process describe above for this installation is fairly reliable.

Students' behaviour changes during these first lessons: at the beginning they have a very school behaviour, questions are mainly adressed to the teacher, they are mainly asking for a definition, a detail, and answers too are adressed to the teacher. After some weeks of practice, they propose conjectures, counter-examples, and some kind of proofs. In some situations they are able to build, within a debate, some collective proof, through conjectures, refutations, transforming statements ... [2].

Some work remain to be done about this installation process: we'll look here at two questions.

-First we still have to study, in a more accurate way, through what evolution the students "coutume" changes from the previous one to the "codidactic" one. Our hypothesis is that this evolution is produced by the negotiation of successive "contracts", each of short time (a lesson or less), between the teacher and the students. These successive contracts are not necessarily totally explicite, nor their negotiation is , so the term of contract is in some sense not very good , but it describes the rules and expectations in use in a collectivity, such as an amphitheatre during a math lesson, as regards the treatment of a definite problem about a definite knowledge. It describes, like the concept of "coutume" some basis of the social interactions in a classroom.[6][7].

- Second we have to carry on the reflection about the following problem: what part of the new "coutume" have to be made clear, and what part not (probably we have to consider the step of the installation process). The problem is that , in some sense, too much expliciting may have some bad consequences of the same kind as those described about proof in the first part, and that we call "contract-effects": to give a proof for the teacher and not because it's a necessary scientific step. Here, to debate, to formulate conjecture for the teacher: students must, in order to build their own knowledge, live some situation-problem in which there is no indication about the didactic intentions of the teacher as regards this knowledge (devolution of problems in a-didactic situations [7]).

Références and bibliography:

[1]Alibert-Grenier-Legrand-Richard: **Introduction du débat scientifique dans un cours de 1ère année du Deug A à l'Université de Grenoble I.** Rapport de l'ATP "Transitions dans le systéme éducatif" Ministère de l'Education Nationale.

[2]Alibert: **Alteration of didactic contract in codidactic situation.** Proceedings of the Eleventh International Conference PME II.p.379-385. Montreal 19-25 July 1987.

[3]Alibert: **Towards a new "coutume" in the classroom.** To appear in For the Learning of Mathematics.

[4]Bautier-Robert: **Apprendre des mathématiques et Comment apprendre des mathématiques.** Cahier de Didactique des mathématiques n°41, Avril 1987. IREM Université Paris 7

[5]Robert: **De quelques spécificités des mathématiques dans l'enseignement post-obligatoire.** Cahier de Didactique des mathématiques n°47, Novembre 1987. IREM Université Paris 7

[6]Balacheff: **Le Contrat et la Coutume, deux registres des interactions didactiques.** Colloque Franco-Allemand de didactique des mathématiques et de l'informatique.Marseille. Novembre 1986.

[7]Brousseau: **Fondements et méthodes de la Didactique des Mathématiques.** Recherche en Didactique des Mathématiques Vol 7-2.1986

THE CONSTRUCTION OF ARITHMETIC STRUCTURES BY A GROUP OF THREE CHILDREN ACROSS THREE TASKS

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The problem solving behavior of a group of three seventh grade children who participated in a five session teaching experiment within a classroom setting is described. The children were given three problem tasks using concrete, nonnumerical embodiments and asked to construct solutions containing common structural elements. Analysis of the problem strategies revealed successful constructions of solutions, recognition of the meaning of the various structures, understanding of the relatedness among representations, and generalization to numerical representations. Contributions and challenges from individuals toward the group activity seemed to facilitate the process.

THEORETICAL FRAMEWORK

Jeeves and Greer (1983) emphasize the importance of developing in children an awareness of the structural relationships in the mathematics that they are using and an ability to recognize structural similarities in situations that appear on the surface to be different. Morris Kline (1974) advocates children's involvement in creating mathematics and proposes that new subject matter be approached through intuitive experiences that could be represented by physical arguments. He contends that children best gain this understanding when mathematics is developed constructively and not deductively.

Resnick and Ford (1981) indicate that research decisions concerning the effectiveness of structural approaches to children's learning must be based on careful consideration of their implementation in classroom situations. They conclude that this evaluation should be based_on precise definition of the mathematical structures to be taught and clear criteria for recognizing what the children come to understand.

Noddings (1985) provides a rationale for the effectiveness of children working in small groups citing the benefits of their encounters of challenges and disbeliefs of peers, the sharing of knowledge, and the provision of responsibility to take charge of their own learning. Previous research of small group problem solving activities indicates that learning does occur for individual members of the group and analysis of the interactions within the group provides rich data for consideration of the process of children's problem solving and their construction of understanding of particular mathematical structures (Alston and Maher, 1984; Maher, Alston, & O'Brien, 1986).

OBJECTIVES

Specific objectives of this investigation were to describe how a particular group of children worked together to solve three problem tasks designed to use a variety of concrete nonnumerical embodiments to construct models of the structure of certain properties of a binary operation on a set of elements: namely, closure, commutativity, identity and inverse elements. The problem solving behaviors that were studied were: (1) construction of solutions based on the representation of the concrete models and/or monitored and revised on the basis of conceptual knowledge; (2) recognition of similarities and/or differences among the tasks; (3) generalizations to numerical representations and (4) individual contributions/challenges to the group problem solving process.

METHODS AND PROCEDURES

Five 45 minute sessions of a seventh grade mathematics class in an independent school were devoted to providing 12 and 13 year old children with an opportunity to construct solutions to three concrete nonnumerical problem tasks (DOLLS TASK, PROBLEM WITH CARDS, AND ROADS TASK) dealing with the structure of the properties of closure, commutativity, identity and inverse. Each of five groups in the class was composed of two or three children chosen by their teacher on the basis of similarity in ability and potential compatibility for working together. Two girls (Tricia and Natasha) and one boy (Ed) who are described in this paper were members of the same mathematics class throughout the year and had been accustomed as a part of regular instruction to working in small groups to solve problems.

The classroom teacher arranged the children into groups explaining that this would be the class context for the activities. A script for each of the problem tasks in turn was given to each of the children along with two sets of the objects appropriate to the task. The children were instructed by the teacher to chose one person to act as official recorder and to have some agreement on the responses recorded. Each child, however, was asked to complete a problem script with his or her own ideas about the solution which might be different. A final section on each task required the children to reflect on the problem solving and asked (1) what they liked and disliked about it, and (2) what other problems or ideas, if any, were called to mind.

The directions concerning each operation were written as a part of the script and the children were asked to demonstrate understanding of the operation. The teacher's role was to respond to questions should clarification of the meaning of each operation be necessary rather than intervening in the children's construction of solutions. The children were permitted as much time as required to complete each problem task and were instructed to return the sheets as each problem was completed before receiving the next set.

Each of the five sessions was videotaped and transcripts of segments of the tapes were obtained by independent viewing by three graduate students. These transcripts along with observers' notes and children's work sheets provided data for the analysis.

THE PROBLEM TASKS

The Dolls Task: Adapted as a group problem solving activity from a clinical interview task to assess students' understanding of the properties of an abelian group, children are given a pair of small figures, boy and girl. The elements of the set are the rotations of these two figures taken together from a facing front position, defined as "Both Turn", "Only Boy Turn", "Only Girl Turn", and "Nobody Turn". The operation on the set is introduced as one rotation followed by a second without returning to the facing front position and the result is the single rotation from a facing front position that would leave the figures in the same final position. The children are first asked to complete a four by four table for the set with the operation and then asked a series of questions about closure, the existence of an identity element and inverse elements, and commutativity of the operation within the set.

The Problem with Cards: A set of five cards, each with a different polygonal shape cut out constitute the elements of the set with the operation defined as putting one card on top of another and the result being the hole formed by the two

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cards. The first part directs the children to use four of the cards and to complete a four by four table for the set with the defined operation. These four cards form a lattice structure that is closed under the operation. The children then are asked to consider a series of questions about closure, the existence of identity and inverse elements, and commutativity within this set. The second part is similar to the first except that a fifth card is introduced and a five by five table is to be completed. However, this set is not closed under the operation.

The Roads Task: This problem task has a script parallel to that described in the Dolls Task but has a cyclic group structure. The members of the set, introduced to the children as Road Cards, are index cards each having lines from four equally spaced beginning points on the left side to corresponding end points on the right. The operation is introduced as one Road Card followed by a second and the result for each pair of Road Cards is the single card having the same beginning and end points for its "Roads" as the beginning points of the first card followed by the final end point reached by tracing along the lines from the first card to the second.

RESULTS

All three of the children successfully completed the chart in each of the problem tasks. Each also responded correctly to the questions concerning closure, successfully explaining under which conditions there would always be a solution within the given set.

The children consistently used the physical objects to model the operation in order to solve this part of each problem. Interaction and discussion among the children regularly occurred as they demonstrated their understanding to each other and challenged each other in their thinking. In each case, before agreeing about the completed chart, each child, using the concrete representation and asking for clarification when necessary, either from one of the other two or an observer, went through each possible combination.

When asked to give reasons about closure, certain comments suggested generalization of the specific understanding. For example, Ed, in defending his explanation of the dolls said that there is always a command that describes the final position because: "no matter what way they face we already have a command for it ... those are the only commands. ... Yesterday we did a number system that had other numbers, so they could have resulted in something else, but it didn't. In this case we don't even have any other so it couldn't even if we wanted it to ".

In filling out the second chart for the "Problem with Cards", Tricia began to say that Card B over Card E results in "1/2 of Card C". Ed challenged her that this would not do because "1/2 of C is not one of the shapes". After all three of the children discussed the issue and reread the original instructions there was agreement that "NONE" had to be the entry into the chart.

It was even more necessary for the children to individually perform each combination of Road Cards in order to understand and complete the operation table for the third problem. However, in responding to the questions on closure for this problem, Natasha argued that her conclusions to each question could be explained by the chart rather than describing the cards.

The children were also successful in responding to the questions about identity and inverse elements within the three problems. In each case, their explanation of why the special command or card was chosen had to do with describing it physically and demonstrating. Tricia showed the other two by using the dolls that Nobody Turns would leave the first position unchanged. In the second problem, her written explanation for the special card as D stated: "Because D is just an empty void - whatever card is on top will cover much of D's space and will not cover the part of D that is needed to make the 1st card's shape"..

Although the children immediately said that the first two problems were alike, no comparison was made between the special card D and the command Nobody Turns. However, in Ed's discussion while figuring out the operation on the Road Cards, he immediately said: "A is the Special Card". Natasha, in explaining A as the special card parallelled her description of Nobody Turns: "Because A is only straight lines and when you add it to another it will result in that card".

In choosing inverse elements, several strategies were used. In filling out the chart for the first problem, the following exchange occurred:

Tricia: "Only Boy Turns and Only Boy Turns. Same as Nobody Turns."
Ed: "Wait - these are all going to be Nobody Turns".

The selection of partner commands for this problem was quickly accomplished. However, in the discussion about an explanation Ed tried to generalize. He first stated: "Any pair of commands cancel out leaving the dolls at Nobody Turns". When Tricia was not satisfied he compared the operation to multiplication of fractions: "1/2 time 2/1. They cancel out." From this point he continued to compare the operation on pairs of inverse elements to fractions.

The three agreed that this was not the case for the Problem with Cards. Their explanation was based on the fact that the Special Card D had the largest opening. Natasha wrote: "Because C is so small that when you place it on top of anything C will always cover some space, so it will never be completely empty" to form D.

In the Road Card Problem, however, all three initially said that each card was its own inverse even though they had already successfully completed the chart. Ed appeared confused in approaching this question, first assuming that it was a restatement of the question of identity. Tricia reread the question and then said: "Let's look at the chart. C followed by C is A". Ed responded: "I know but it doesn't make sense". To which Tricia responded by demonstrating with the cards. Ed agreed and then generalized: "Because two of any card give straight lines". He entered the four cards as their own partners then looked at the chart and corrected himself: "It's just A and A and C and C. Not all of the cards work like that". The three agreed and corrected their papers.

The students had a general discussion during the first problem describing the patterns in the chart. They described the symmetry of the table but did not refer to this in answering questions about order. In the first problem the three seemed to misunderstand the general question about order, assuming still that it is referred to pairs of inverse elements. Early in the Problem with Cards, however, Ed commented that the operation was commutative. The other two agreed and used this as a reason to justify all of the questions dealing with order. Similarly, after completing the chart for the Road Cards, Ed stated again that the operation was commutative and used this knowledge to explain the question of order for the identity element.

The questions asking for explanations about which they had to agree led to increasingly articulate descriptions of what was happening in each operation. The variety of representations in which these properties were considered seemed as Ed said: "to show us to look at problems from a different point of view". They stated that the most important understanding in solving the problems was understanding the commutative property and likened these problems to addition, multiplication and reciprocals of fractions.

The value of the same group of children working together over a period of time seemed particularly evident in following Natasha, who was the least assertive of the three. During the first problem task, although she participated by following along, using the figures and responding in writing to each question, Natasha did not take an active part in discussion. Increasingly, during the other two tasks, she became more involved and was central to the discussion on the Road Card problem, showing the others that the chart was key to understanding the identity and inverse relationships.

CONCLUSIONS AND IMPLICATIONS

The investigation provided an analysis of the mathematical interaction among three children working over a five day period in a regular classroom setting on a series of non-routine mathematical tasks. Findings suggest that the activities provided an opportunity for the children to build cognitive structures by their actions on the objects that were provided. Understanding of the concepts inherent in the tasks was enhanced by the group problem solving activities. There were various strategies used by the children in constructing solutions to the task. Further analysis is required to determine the nature and extent of the interaction among members of other groups who participated in the activities.

That there was no direct teaching during this period suggests that it is possible to design learning activities for 12 and 13 year old children that promote the construction of multiple representations of mathematical ideas. Further work is needed in considering problem tasks appropriate for a variety of mathematical concepts and a range of cognitive abilities of children. The appropriateness of these activities as a regular part of instruction deserves serious consideration for those who are challenged to consider constructivist approaches to teaching and learning.

REFERENCES

Alston, A. and Maher, C. A. (1984). Mathematical structures and problem solving. in H. D. Thornburg, (Ed.) Middle School Research: Selected Studies. Columbus, Ohio: National Middle School Association.

Jeeves, M. A. and Greer, B. (1983). Analysis of Structural Learning. New York: Academic Press.

Kline, Morris (1974). Why Johnny Can't Add: The Failure of the New Math. New York: Vintage Books.

Maher, C., Alston, A., and O'Brien, J. (1986). Examining the heuristic processes of nine to twelve-year old children in small group problem-solving sessions. Procs. of the Tenth International Conference of Psychology of Mathematics Education, 369-374, London, England: University of London Institute of Education.

Noddings, Nel (1985). Small groups as a setting for research on mathematical problem solving. In E. A. Silver (Ed.), Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives. Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Resnick, Lauren and W. W. Ford (1981). The Psychology of Mathematics for Instruction. Hillsdale, N. J.: Lawrence Erlbaum Associates.

CAREER CHOICE ,GENDER AND ATTRIBUTION PATTERNS OF SUCCESS AND FAILURE IN MATHEMATICS.

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Self image and confidence in personal ability in mathematics, as shown in the causes one attributes to their successes and failures in high school mathematics, may affect one's future behavior. This study examined whether there is a connection between attribution patterns, gender and career choice.
First year university students (135 females, 166 males), belonging to five career curricula which demand different amounts of mathematics, were given the "Math Attribution Scale".
Results indicated that : 1) In the population as a whole, females attributed their success in mathematics to occasional and external factors such as effort extended or nature of task, while males attributed their success in mathematics to a constant and internal factor such as personal ability. 2) Within a career curriculum student - regardless of gender, attributed the same causes for their successes and failures in mathematics. 3) Between the different career curricula, significant differences were found in all varibles of causal attribution of performance in mathematics. These results are highly correlated with the amount of mathematics requiered in each career choice.

Introduction :

Research on gender differences in highschool mathematics has indicated that the relative percent of females choosing to study advanced mathematics is significantly lower than that for males (Carpenter 1983, Cockroft 1982). In Israel about 27% of high school males choose to study high level courses in mathematics, while for females only 12% study at this level. (Israel Central Bureau of Statistics, 1985).
Reasons for the differences mentioned above are :
1. Families' and peer groups' perception; mathematics is considered as a male domain. Women do not intend to choose careers in science and technology, and therefore they find no need for higher level mathematics.(Fox 1976, Armstrong, Price 1982).
2. Females more then males, suffer from "Math anxiety. This anxiety is seen as a major factor for the reduced rate of females in higher level mathematics (Tobias 1978).

3. Females have a low "self`image" and lack of confidence in their personal ability to handle mathematics. This is not to say that male do not experience such feelings. They do but, females attribute their short comings to different factors than do

males. High school females attribute their success in mathematics to unstable or external causes: (I´m successful in mathematics because of good learning environment, easy tasks, effort extended etc.).While their failures in mathematics are attributed to internal and stable factors, such as the lack of personal "mathematical" ability. Males, on the other hand, attribute their success in mathematics to personal ability and attribute their failurs to external and occasional factors. (Wolleat et al 1980, Amit & Movshovitz 1987).
Causal Attribution Theory (Weiner, 1974), states that causes for success and failure in performance can be attributed to: 1. personal ability 2. nature of task. 3. effort extended. 4. luck or environment. These attributions are considered as being external or internal, stable or unstable, as shown in the following table.

Causes	Internal	External
Stable	Ability	Task
Unstable	Effort	Environment

According to Causal Attribution Theory, a person´s perception of their own success and failure, influences their future behavior. For example, one who attributes his successes or failures in mathematics to an internal and stable factor, such as ability, expects this success or failure pattern to repeat itself in the future. But it seems reasonable to assume that one who attributes his performance in mathematics to external and occasional factors, believes that these patterns of performance are not necessarily repeated. Moreover, it can be assumed that if a person attributes his successes to internal and stable factors, and his failures to unstable and external factors, then this person may have a high and positive self image about his mathematics ability. On the other hand, if a person attributes failure to internal and stable factors, and success to external and unstable factors, it seems that the person has a low self image in his mathematical ability.

This study examines whether or not Causal Attribution Theory applies at the university level, where students have already made career choices which require different amounts of mathematics.

Objectives

The research questions for the study were:

1. Are there gender differences within the university population as a whole in one´s perception of their successes and failures from high school mathematics.(In Israel one starts university 3 to 6 years after graduation from high school).

2. Within a particular career choice, are there gender differences in one's perception of their successes and failures from high school mathematics.

3. Are students in different career curricula different in their perception of their successes and failures from high school mathematics.

Research design, population and instrument :

301 first year university students (135 females, 160 males), ages 21-25, belonging to five different career curicula. Humanities (27 f. 22 m.), Education (32 f.), Biology (37 f. 37 m.),Technology (27 f. 47 m.), Math/Physics (18 f. 60 m.), were given the "Math Attribution Scale" questionare. This scale, based upon C.A.Theory, was developed and tested for reliability and validity by Fennema et al (1979) and adapted by Amit et al (1987). This instrument has been used in previous research studies (mentioned above) and seems to be the best measure available for assesing causal attribution characteristics in mathematics.
From analyzing and scoring the questionnaire, eight attribution patterns evolve:

1. success, ability.
2. success, effort.
3. success, task.
4. success, enviroment.
5. failure, ability.
6. failure, effort.
7. failure, task.
8. failure, envir.

Main results

For each of the eight attribution
patterns and for every student, a mean attribution score was calculated. (range of scores 4-20).T-tests, used to establish significant gender differences on the eight attribution patterns. ANOVA on the factors of attribution patterns, gender and career curiculum were analyzed.

1. In the population as a whole, significant gender differences in percieved causes of success were found . (fig. 1)

As observed, females, more than males, attributed their success in mathematics to external and unstable causes (e.g. ease of task, a supportive learninig environment and effort extended). On the other hand, males significantly more than females, attributed their successes in mathematics to personal ability, which is an internal and stable factor.

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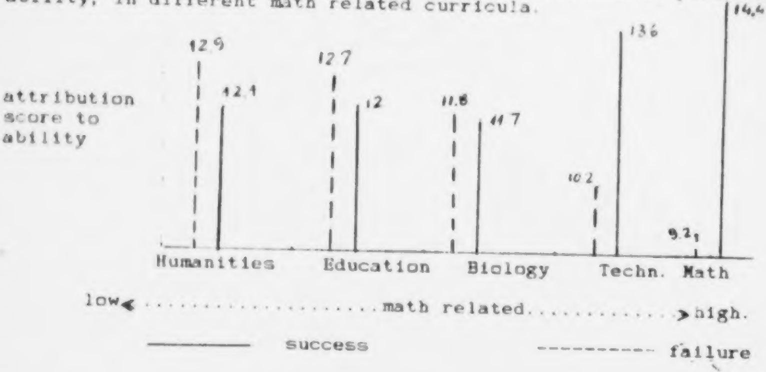
Figure 1 : Attribution scores of success and failure in mathematics. (mean scores, S.D)

	Success				Failure			
	Ability	Effort	Task	Envi.	Ability	Effort	Task	Envi.
female	12.4 3.1	15.1 2.9	15.1 2.5	16.4 2.3	11.3 3.6	15.3 2.8	13.1 2.9	11.3 2.9
male	13.3 3.3	13.9 3.8	14.4 2.7	15.1 2.9	10.7 3.9	15.1 3.0	12.5 3.0	11.6 2.9
Sign $p<.05$	*	*	*	*				

2. Within the same career curriculum, no gender differences were found in one's perception of success and failure. In other words; students; regardless of gender, choosing a common curriculum, attribute causes for their successes and failures in the same way.

3. Focusing on the scores between the different career curricula, statistically significant differences were found in all variables of causal attribution. The mean scores attributing success and failure in mathematics to personal ability is shown in Figure 2.

Figure 2 : Attribution of success and Failure to personal ability, in different math related curricula.



As observed in figure 2, the more mathematics demanded in a curriculum, the more students in this track attribut their success to personal ability. On the other hand, the less mathematics demanded by the curriculum, the more students attribut their failures to this internal and stable factor.

Discussion:

The main results show that within a career curriculum, gender differences in causal attribution of success and failure in mathematics does not appear. Between different career curricula, strong differences in causal attribution of success and failure in mathematics do surface, and are correlated with the amount of mathematics the career curriculum demands.

The results of this study are supportive of Causal Attribution Theory; namely, when one attribute their successes in mathematics to internal factors, there is a high likelihood of studying a math-related curriculum (Fig. 2, Technology and Math/Physics). Similarly, when one attribute their failure in mathematics to internal factors, there is a high likelihood of not studying a curriculum with high mathematics requirements. (Fig. 2, Humanities and Education).

Results of this study also support the previous research with high school populations, which shows significant differences between students studying in low demanding and high demanding mathematics courses. However, in this privious research, significant gender differences did occur within each high school course of study. Here, in this research, those gender differences did not appear within each career curriculum track. This we attribute to a selection process which is based upon self perception of ability and success. In high school there are some females with high mathematics self-image. They are the few who choose a mathematics related curriculum in the university. Moreover, they choose these university curricula because of their self- image, independent of their high school grades. (previous research show that males and females are essentially equal in cognitive achievement).

The importance of this result can not be over stated. This study showes that attribution patterns which are developed in high school, are relatively permanent and affect future career choice. Therefore, high school teachers and counsellors must take special measures to develop a positive self-image with their female students, in order to enable them to pursue math related careers in the future.

Bibiliography :

- Amit m. Moushovitz-Hadar n.,"Sex Differences in Causal Attribution in High school mathematics". Paper presented at P.M.E XI, Montreal, 1987.

- Armstrong ,J.M, Price R.A "Correlates and Predictors of Women's Math Participation". Journal for Reasearch in Mathematics Education. 3 pp. 99-109, 1982.
- Carpenter T.P. Lindquist M.M, Mathematics, W. Silver, E.A. "Results of the Third NAEP Mathematics Asseosment Secondary School". Mathematics Teacher, 12 pp. 652-659, 1983.
- Cockroft, W.H. "mathematics Counts". Report of the Committee of Inquiry into the Teaching of Mathematics in Schools. London; Majesty's Stationary office, January 1982.
- Fennema, E., Wolleat, P. Pedro J.D. "Mathematics Attribution Scale". Journal Supplemental Abstract Service, May 1979.
- Fox, L.H. "Women and the Career Relevance of Mathematics and Science". School Science and Mathematics 26.pp 347-353, 1976.
- Toias, S. "Overcoming Math Anxiety" New-York, Norton, 1978.
- Wiener, D. "Morristown N.J. General Learning Press. 1974.
- Woleat, P.L., Becker, A.V., Pedro J.D, Fenneme, E. "Sex Differences in High school Student's Causal Attribution of Performance in Math. Journal for Research in Mathematics Education . 11, pp. 356-366 1980.
- C.B.S - Israel Central Bureau of Statistics; "Distribution of Sex by Mathematics Learning Level in High school", 198[illegible].

A CLASSIFICATION OF STUDENTS' ERRORS IN SECONDARY LEVEL ALGEBRA

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Errors made by seventh to ninth graders in classroom tests, homework, or during lessons, are analyzed and classified according to a catalogue of forms of "illegal thinking", based upon a theoretical model describing problem solving processes as a sequence of states generated by applying "operators". The following types of "illegal thinking" are illustrated by examples:
(1) Erroneous application of an operator
(2) Erroneous execution of an operator
(3) Lack of identification or erroneous matching of variables in the premise of an operator
(4) Incorrect identification of variables in the conclusion .

Information processing approaches are increasingly used to analyze and to classify students' errors in topic areas of school curriculum. Within a framework of this kind, a problem or a non routine task is described by three components: (1) a starting situation or initial state, defined by the givens, (2) a final situation or final state to be attained by the problem solution, (3) a barrier which at the beginning prevents immediate transition from the initial state to the final state. The transition is described as a sequence of intermediate states, which can be obtained by applying appropriate operators to transform the preceeding states.

Operators are characterized by a premise and a conclusion; their application demands to check, whether the premise of the operator fits to the state to be transformed, and yields the conclusion of the operator as the subsequent state, when the variables occurring in both the premise and the conclusion are replaced by names of suitable objects.

Classroom tasks and problems given to seventh to ninth graders are mostly of the "interpolation barrier" type, which means that goal criteria are clear to the students and the means well-defined (the repertory of means closed); there might as well be problems with indistinct goal criteria or with an open repertory of means. (Cf. Doerner, 1979)

In many situations and by different reasons problem solvers use "illegal operators" instead of making correct use of an operator, for instance they omit complete checking of the applicability of an operator, or use modified operators formed by themselves. In his basic model, D. Doerner has established a catalogue of illegal operators used by university students trying to deduce theorems in propositional calculus. (Doerner, 1973). This catalogue has to be refined when being used in the context of more complex problem types, as compared with propositional calculus.

There is another difference between the conditions under which Doerner's university students solved logical problems, and the situation in classroom: Doerner gave his volunteers a list of "axioms", whereas in classroom students will have to use means to be retrieved from their memories. It cannot be excluded that frequency and distribution of error types depend on situational components, such as just mentioned; but it turned out that the error categories of the modified catalogue occurred in quite different topic areas of school curriculum.

The present context deals with errors made by seventh to ninth graders in classroom tests, homework or during lessons, in "elementary algebra", i.e.: linear equations and inequalities, use of binomial formulae, and quadratic equations. Objects or states, to be transformed by operators, in this special case are the left sides of equations, when any formula is used in order f.i. to simplify a term, equations or inequalities, when their solution set is determined by applying any transformation rule. Students' errors in this field are analyzed and classified according to the following list of "illegal operators":

(1) Erroneous application of an operator

(2) Erroneous execution of an operator

(3) Lack of identification or erroneous matching of variables in the premise of an operator

(4) Incorrect identification of variables in the conclusion.

The types (1) and (2) correspond to the types (NCON) and (NEX) in Doerner's catalogue respectively, which are described as non-consideration of the conditions for the application of an operator, and non-consideration of the application instructions. Within non-consideration of conditions there has to be made a difference between checking whether the conditions of an operator fit to a special situation itself, and whether the user makes a correct matching of the variables occurring in the premise of an operator to the terms substituted for these variables, i.e. finding a correct instantiation for the variables of the operator. The same holds for Doerner's type (NEX), with respect to the conclusion of an operator. The types (3) and (4) of the present catalogue do not occur in Doerner's list; as can be seen from the tasks Doerner gave his students, there are only a few variables at all in these tasks, such that erroneous instantiation actually can be excluded. But as soon as students have to carry out substitutions for variables in a formula - which is an important partial aspect in algebraic skills - f.i. when applying binomial formulae or transforming inequalities, the consistant and correct use of the same term for any variable is crucial. So, in more complex topics in school curriculum, such as algebra, or even more geometry,

these two components have to be separated distinctly from one another.
Doerner's catalogue still contains further types of "illegal thinking":
Invention of new illegal operators by analogy transfer (AN)
Invention of new illegal operators by "semantic" considerations (SEM)
Invention of new illegal operators par force (PAR)
Search for external causes for the "unsolvability" of a problem (EXT).
They can be neglected in the present context, because they either are specific for the situation or the topic area from which Doerner obtains them, or cannot be sharply separated from (NCON) and (NEX) when used to describe errors in school algebra.

The examples given below do not make any assumption about whether a modified operator has been learned instead of a correct one, or whether an operator correctly retrieved from memory is applied in an erroneous way due to any reason determined by the special situation. The latter could be made sure, if f.i. the student would be able to give a correct answer when asked for the transformation he or she just carried out.
Usually it seems to be impossible to ascertain which explanation is conclusive. Mostly the special situation is not appropriate to investigate the actual root of the erroneous strategy. But even in situations where it is possible to ask the proband to explain or to comment the use of a certain transformation, he or she will answer something like: "I just did it in that way", "Because I have to solve the equation", "In order to continue", "I did it in the same way as in the previous example", "Isn't that correct?"

In transferring Doerner's catalogue to any more complex topic area than the originally used propositional calculus, we have to take into consideration a fifth type of illegal operator neither enumerated in his list. In principle it can be imaginated that illegal operators of the following type are used:
(5) confusing premise and conclusion of an operator.
In fact, this error type can be established in geometrical proof problems (cf. Becker, 1986), but in elementary algebra tasks it was not yet to be found. Algebraic transformations are mostly of the equivalence type, such that confusing premise and conclusion is of no relevance in the present context. Since solving inequalities needs the use of transformations in which premise and conclusion are well separated from one another (f.i. If $a < b$ and $c > 0$, then $a \cdot c < b \cdot c$), we could expect errors of type (5) to occur. However, the material available from classroom work is so restricted that there was no opportunity to find evidence for illegal operators of this type.

(1) Erroneous application of an operator

This error type is characterized by neglecting the conditions for the application of an operator, i.e. a given combination of symbols is misunderstood in such a way as if allowing a transformation which actually cannot be carried out when starting from the given pattern. Thus, the domain of application of the operator is extended, which is an overgeneralization with respect to the premise of the operator.

The well-known cases, in which a reduction of a fraction "out of a sum" takes place, belong to this type, such as

1.1 $$\frac{-a^3 + a\cancel{b^2}}{\cancel{b^2} - a} = \frac{-a^3 + a}{-a}$$

or

1.2 $$\ldots = \frac{x^2 + \cancel{y^2}}{y^2 - \cancel{x^2}} \cdot \frac{\cancel{x^2} - y^2}{-x^2 + \cancel{y^2}} = -\frac{\cancel{y^2} + \cancel{y^2}}{\cancel{y^2} + \cancel{y^2}} = -1$$

Certain transformations of algebraic terms are based upon properties of the number 0; we often find erroneous transformations formed by overgeneralization of those properties, consisting in transferring them to other specialized numbers, such as 1. The same holds for the converse direction. Correspondingly transfer from addition towards subtraction or multiplication takes place, and from multiplication to "related" operations including raising to a power, which leads to formation of illegal operators of type (1).

Erroneous operators of this type are not at all restricted to terms, but occur in solving equations or inequalities as well, such as in

1.3: The solution set for the following system of equations

$$\begin{aligned} 0.3\,x + 1.5\,y - 9 &= 0 \\ 2\,x + \;\; y \quad &= 6 \end{aligned}$$

after several intermediate steps, among these

$$\begin{aligned} 3\,y - 18 &= -0.6\,x \\ 3\,y - 18 &= -6\,x \end{aligned}$$

is determined as the empty set; the argument for this result was the hint at the equal coefficients 3 and -18 on the left side in contrast to the different numbers -0.6 and -6 on the right side, with reference to the same result for the system

$$\begin{aligned} a\,x + b\,y + c &= 0 \\ a\,x + b\,y + c' &= 0 \qquad \text{for } c \neq c' \end{aligned}$$

A series of errors generated by unsufficient checking of the premise may be explained by disregarding any convention, which also can be interpreted as "extending" a convention, among these disregarding of parentheses rules or rules concerning the order of succession of different operations, especially the numerous cases of breaking the distributive law.

Conventions are disregarded, too, when symbols for an operation are omitted, or omitted symbols inserted in a faulse way, as was done in

1.4 $(3\frac{1}{3} - 4)(3\frac{1}{3} - 3) = (3 \cdot \frac{1}{3} \cdot - 4)(\cancel{3} \cdot \frac{1}{\cancel{3}} \cdot 3) = \ldots$

The following example can be explained by the influence of verbal labels, which can be described as "changing twice" with the expectation that both changes together cancel each other:

1.5 $-3x > -1$ is regarded as equivalent to $x > -\frac{1}{3}$

1.6 $-4x < -9$ as equivalent to $x < -2\frac{1}{4}$

where the change in the order sign (by multiplying with a negative number) is thought to be unnecessary, because the negative sign (on the right side) is omitted simultaneously. The intermediate line is not written down, purposely and with the explicit explanation, that a correct line is not obtained before both changes are performed.

"Cancelling each other" is a commonly given argument for transformation steps having as origin partial terms which actually would cancel each other in quite another context, f.i. as inverse elements with respect to any operation:

1.7 $\ldots = -6a \cdot a + 6 \cdot a = a \cdot a \cdot a$

Here the two numeral factors $+6$ and -6 are obviously imaginated as cancelling each other (when added), the remaining factors are combined to the product. Similarly, addition of $+3$ is thought to give reasons for the following transformation

1.8 $-3x > -14$ into $x > 3 - 14$

which again shows the confusion of addition and multiplication.

(2) Erroneous execution of an operator

This error type consists in modifying the conclusion of an operator instead of processing a term according to the correct use of the operator, and thus extending the domain obtainable by applying the operator. Very often erroneous modifications of the conclusion of an operator are recognizable as incorrect generalizations disregarding specific properties of different operations.
For instance

2.1 $\ldots 3 \cdot \frac{4}{9} p^2 = \ldots 1\frac{3}{9} p^6$

reveals an extension of the distributive law (pertaining to exponents), similarly as in

2.2 $-2\frac{6}{18} p^2 + 1\frac{6}{18} p^6 = -1 p^8$

with different kinds of "comprising" partial terms; and so do the following:

2.3 $2(2 + 3x) - 1 + 2x = 4 - 1 + 5x$.

2.4 $\frac{63}{12} xy + \frac{42}{3} x^2 y \ldots = \frac{105}{36} x^3 y^2$.

the last example showing the nominators of numeral factors being added. as usual, the denominators multiplied, as in multiplication of fractions. the exponents in the variable factors again added.
The next example,

2.5 $$\frac{7}{5}bc \cdot \left(-\frac{2}{7}ac\right) \cdot \frac{19}{8}abc = -\frac{2}{5}c \cdot ab \cdot \frac{19}{8}abc = 2ab \cdot \left(-\frac{19}{40}c\right)$$

consists in bringing common partial factors in every two factors out of - imaginated - parentheses: c in the first step, ab in the second step, and again c in the last step.

The "generalized distributive law" (referring to multiplying two sums in parentheses), with the binomial formulae as special cases, gives rise to a variety of erroneous applications especially of type (2), such as

2.6 $$\left(\frac{7}{3}x - 8\right)\left(\frac{9}{4}y + 6xy\right) = \frac{63}{12}xy - 48xy$$

2.7 $$(5r + 4ps)(4ps - 5r) = 16(ps)^2 - 25r^2 + 40psr$$

2.8 $$\left(2\frac{1}{2}x - \frac{2}{5}y\right)^2 = \frac{25}{4}x^2 - 2xy - \frac{4}{25}y^2$$

2.9 $$\left(\frac{1}{9} + x^2\right) = \left(\frac{1}{3} + x\right)\left(\frac{1}{3} + x\right)$$

In these examples it is easy to be seen, which correct formula was generalized, an in which erroneous way.

(3) Lack of identification or erroneous matching of variables in the premise of an operator

Committing an error of this type means that a number, a variable, or a combination of both is incorrectly substituted for a variable as part of a formula, when applying the latter from one step of a transformation to the subsequent step (usually from the left side of an equation to the right side). Incorrect substitution generally is incomplete or inconsistent substitution: f.i. if the same variable occurring in the premise is not constantly substituted by a special number or by the same variable, or if the product term $(5x - 3)(5y - 3)$ is transformed by using a binomial formula, or if the partial terms (summands) in $(5x - 3)(3 - 5x)$ in the same order as written down are substituted for the variables a and b in $(a - b)(a - b)$ (thus $5x$ for a, 3 for b, then 3 for a and $5x$ for b
In certain cases this matching cannot be carried out before having performed necessary transformation steps, f.i. when deciding, whether a sum is a quadratic term, such as in

3.1 $$16b^2 + 80bc + 25c^2 = (4b + 5c)^2$$

and

3.2 $$16b^2 + 80bc + 25c^2 = (8b + 5c)^2$$

Confusion of doubling with squaring (both is "doing the same twice") often leads to erroneous instantiation (in the premise), such as in

3.3 $$z^4 - \frac{1}{256} = (z^2 - \frac{1}{128})(z^2 + \frac{1}{128})$$

Errors of this type will have as consequence obviously an error of the subsequent type.

(4) Incorrect identification of variables in the conclusion.

This error type corresponds to (3), but mismatching takes place in the conclusion of the operator. It may pertain to a numerical factor, to a variable factor, or to a combination of both:

4.1 $$(4\,uw - 3\,v)(3\,v + 4\,uw) = 16\,uw^2 - 9\,v^2$$

4.2 $$(\frac{5}{6}\,abc - \frac{4}{9}c^2d)(\frac{5}{6}\,abc + \frac{4}{9}c^2d) = \frac{25}{36}\,ab^2 - \frac{16}{81}c^4d$$

4.3 $$(4\,uw - 3\,v)(3\,v + 4\,uw) = 4\,uw^2 - 3\,v^2$$

Especially when writing down all intermediate steps referring to this very matching process, errors of this kind can be well identified, such as in the following solution of a task, which consists in finding the third summand to fit to the first two summands, such as to form a quadratic term:

$4\,y^2 - 6\,xy + \ldots = (\ \ldots\ )^2$ As a solution there is given

4.4 $$\begin{array}{l|l} b^2 & \\ a^2 & 4\,y^2 \\ a & 2\,y \\ 2\,ab & 6\,xy = 2 \cdot 2\,y \cdot 1.5\,x \\ b & 1.5\,x \\ b^2 & 3\,x^2 \end{array}$$

in another case

4.5 $$\begin{array}{l|l} & \vdots \\ 2\,ab & 6\,xy \\ b^2 & 2\,x \end{array}$$

Obviously in the two last examples the unspecified labelling "twice the same" again played a role in causing the erroneous procedure.

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Further problems and investigations

The main purpose of this paper is to demonstrate that the sketched model of problem solving processes is an appropriate means to systematize and to analyze students' errors in a well-defined topic area, such as elementary algebra; illustrated by examples of the mentioned topic area, the categories can be established in other topic areas, too. Frequency and distribution of the different error categories obviously depend on situational components of the instructional context; it therefore could be a reasonable goal in future work to characterize special situations by the frequency of the different error types. Far more important seems to be trying to investigate the influence of language on the formation of errors, in a double sense: when forming illegal operators and storing them in memory, and when retrieving correctly formed operators from memory and using them in an erroneous way.

Up to now, there is not yet enough material to devise a precise design for investigating these phenomena, the available material stemming from situations which are too inhomogenious: different classes, different instructional conditions, different kinds of registration.

Quite another direction in which the present work can be continued, is the pedagogical aspect, under which the findings to hand could be exploited. In an apparent way analyzing students' errors can be used to devise instructional methods in order to help individual students to eliminate deficits, or to design parts of lessons. Dissolving and phasing the complex course of thinking processes, even if hypothetical, may be a suitable help in designing sequences of steps in instruction.

References

Becker, G. (1986). "Illegal Thinking" in Solving Geometric Proof Problems. In: Proceedings of the Tenth International Conference. Psychology of Mathematics Education. London (375 - 379)

Becker, G. (1987). Die Bedeutung von Lern und Denkmodellen der Kognitionspsychologie für den Mathematikunterricht - aufgezeigt an Beispielen aus dem Algebraunterricht der gymnasialen Mittelstufe. Der Mathematikunterricht, 33, Issue 1 (5 - 20)

Doerner, D. (1973). Illegal Thinking. In Elithorn, A., Jones, D. (Ed.), Artificial and Human Thinking. Amsterdam. Elsevier (310 - 318)

Doerner, D. (1979). Problemloesen als Informationsverarbeitung. Stuttgart. Kohlhammer. 2. ed.

TEACHERS' WRITTEN EXPLANATIONS TO PUPILS ABOUT ALGEBRA

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This study investigates some aspects of the written language used by teachers to explain to pupils various basic algebraic ideas. The sample consists of 40 student teachers attending a one year teacher training course in mathematics (18 female and 22 male). The origin of the research is an ongoing study of how pupils explain in writing to their peers some aspects of algebra, thus permitting the examination of the relationship between pupils' written language and their understanding of algebra. The pupil questionnaire was modified for the student-teachers, using essentially the same content but asking the student-teachers to imagine explaining in writing the algebraic ideas to pupils. All subjects were given 6 questions. Because of the limited space the results of only 3 questions are reported however the student-teachers' written language reveals a number of misconceptions in the areas of factorise/product, formula/equation, and ratio.

INTRODUCTION AND METHOD

Mathematics and specifically algebra is one of the subjects where, because of its nature, the use by teachers of certain types of linguistic forms could have important implications for children's learning since the teachers' language might well constitute barriers to understanding. Austin and Howson (1979) argue:

> "Yet the use of words to communicate accurately, and in Thom's sense 'with meaning', the abstractions of mathematics is a difficult, perhaps impossible task." (p. 178)

The language used in teaching algebra is necessarily full of symbolism. However while a greater use of symbolism could lead to an increase in the pupils' competence to manipulate symbols it could also simultaneously lead to a decrease in their ability to understand the underlying meanings. Howson (1980) argues that recent reforms have emphasised both the introduction and the use of new symbols and language with the idea that this would enhance understanding and add precision but, he also adds : "(the language) was used so loosely and erroneously that the position is now worse than it was before." (p.571). It is, therefore, crucial to assess the extent to which either the technical words or the ordinary everyday language substituting for these terms help or hinder communication of algebraic ideas in the classroom.

However, commenting on pupils' language - which to some extent will reflect the teaching - Austin (1973) suggests that often teachers both cannot cope with and also fail to recognise the validity of "formulations of ideas which are not expressed in the subject register". The idea for the present study stemmed from a larger study, presently in progress, that looks at pupils' "writings" about algebra, where we are examining, through an analysis of pupils' written language, their understanding of the terminology, the conventions and the 'rules' used in algebra. The questionnaire presents pupils between the ages of 14 and 16 with 6 to 8 situations, and in each instance fictional pupils are shown as having some problems with relatively simple algebraic ideas and subjects are asked to imagine explaining, in writing, a solution that will help the fictional pupils.

It occurred to us that it would be sensible to ask a group of teachers to do the task, and to compare the responses of both groups. Thus the present study examines, by means of a questionnaire, the written language of a group of 40 student-teachers, pursuing a Post Graduate Certificate of Education course in mathematics. The 6 core questions given to pupils were reformulated, using essentially the same content but asking student-teachers to imagine themselves with the task of explaining in writing different aspects these algebraic problems to pupils of a specified secondary school year. The task took about half an hour.

At first we had hoped to use for the student-teachers' explanations some of the more general categories, presently being developed for the pupils' responses so as to make comparison more easily. But the data from the student-teachers does not lend itself to such a classification. The student-teacher (now referred to as 'students') study is reported in its own right, and at a later date a comparison of the two studies will be made. Due to lack of space, the results of only 3 of the 6 questions are presented.

QUESTION I

In this question we are interested in seeing how teachers explain different aspects of algebra, for example, the meaning of the convention "/" in the expression x/s as used in arithmetic and algebra: and the role of variables in expressions like x/s or b:c. Students were asked to imagine that they were explaining to a pupil whether or not the following pairs of expressions 'tell us to do the *same thing* or *different things*': i) 3/4 and x/s; ii) 3/4 and b:c; and iii) x/s and b:c.

A number of students did not answer all three parts: 4, 5 and 10 for parts i, ii and iii respectively; a second different group of students also did not answer the question but instead gave a general pedagogic commentary, for example, about the role of the context of the question; there were 4, 5 and 5 students in the 3 parts respectively. If lack of response can be interpreted as an expression of difficulty then part iii was problematic for more than a third of the students.

Results about judgments: we had expected that in part i) students might state that the two expressions could 'do similar things' while qualifying in which domain, whereas in parts ii) and iii) we expected that students would state that the expressions were 'telling us to different things.' Our expectation was mainly confirmed in first part but none specified domains. However, the responses for parts ii and iii were more problematic: in part ii 6 students judged the 2 expressions as 'doing the same thing' and 12 avoided judgment, in part iii) 10 judged them 'as doing the same thing' and 5 avoided judgment.

Turning to explanations, students did not explain why the expressions were doing 'similar or different things', but instead described separately each of the two expressions. Explanations had both a form and a content. First, concerning the form of explanations, it became apparent that students either "explained" or started an explanation by *naming*, for example, "that is a division"(see Table 1). The content of the explanations presented two sets of issues: i) the interpretation of x/s, and 3/4 either as fraction or as division; ii) the nature of students' explanations for ratio.

Table 1: Form of explanations

	Total	Subtotal		Naming		Name & Explain		Explain		Others
		1st	2nd	1st	2nd	1st	2nd	1st	2nd	
Part i	40	32	31	1	1	3	3	28	27	8 (4 N.R)
Part ii	40	29	30	6	5	6	12	17	13	10 (5 N.R)
Part iii	40	25	24	7	6	5	8	13	10	15 (10N.R)

N.R. stands for non-response

Table 1 shows that for part i the predominant type of response is that of *explaining* rather than *naming*, with approximately three-quarters of the students explaining both expressions in terms of ' division'. In part ii the response is very different, with the strategy of *naming* or *naming and explaining* being used by 12 teachers for the first expression and by 17 teachers for the second. Explanations for the first expression were divided equally between division and fraction.

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In part iii approximately half the teachers replying used the strategy of *naming* or *naming and explaining* for both expressions, with an 11:8 ratio of division to fraction explanations for the first expression.

It could be argued that the expression seen as representing a division does not requiring *naming* since pupils have been familiar with this operation since primary school and so the focus is on explanation. For fractions which are perhaps less familiar the incidence of *naming* increases though not as much as for ratio which could be considered the least well known expression. In both cases the incidence of *explanation* decreases and this could be interpreted as *naming* taking on the function of explaining in some instances.

Four categories were developed to classify students' explanations of ratio (numbers of responses are indicated in brackets) : (a) Sufficient or acceptable descriptions of ratio (parts ii: 7; iii: 5); (b) responses in terms of relation of parts to whole, for example, "a ratio is when the whole is divided into two parts, b and c, and b+c gives the totality or the whole" (parts ii: 7; iii: 4); (c) inexplicit or problematic responses, that is, when it was extremely difficult to interpret what the respondent meant (parts ii: 5; iii: 6); (d) misconceptions (parts ii: 12; iii: 11) ; Others (ii: 9; iii: 14).

There were only very few students in parts ii or iii who gave acceptable or sufficient explanations of ratio. (7/31 and 5/26). The incidence of what we considered as misconceptions is fairly high and three examples of those most frequently found are given below:

- "Fractions and ratios are interchangeable, here 3/4 can be written as 3:4 and similarly b:c can be written as b/c";
- "b:c is another way of writing b/b+c so we must see if we can write 3/4 in another way as b/b +c, we choose numbers for b and c so that b/b+c is the same thing as 3/4"
- "when you divide one number by a second number and have a fraction for an answer it means that there is a ratio between the two numbers, a ratio is shown by the two dots :"

QUESTION 2:

Some words and expressions are used together frequently in algebra, for example, *multiply* and *bracket* in : "when you remove the brackets you must *multiply* everything inside the *bracket* by what is outside the *bracket*. By presenting words or expressions in pairs, specific contexts of algebra can be given a focus. We wanted to see whether or not each pair was identified as belonging to a particular context and how students explained this. So students were presented with the following five pairs and asked " to explain to pupils why the terms (in each pair) *do* or *do not* go well together" : (a) algebraic expression - terms; (b) formula - substitute; (c) multiply - bracket; (d) factorise - product; (e) equation - formula.

We examined first the total response to each part of the question which was as follows: a) 87%; b) 85%; c) 83%; d) 65%; e)68%; so, again , if absence of response could be an indication of difficulty *factorise* and *product*, and *formula* and *equation* were more problematic. We then examined whether or not students thought that the words in each of the pairs could be grouped, and this is shown as a ratio of 'for '/'against' grouping as follows: (a) 32 : 3; (b) 28: 6; (c) 18 : 15; (d) 20 : 6; (e) 22 : 5. So the majority of students perceived four of the five pairs as being able to be grouped together. Only the terms *multiply - bracket* were perceived as not having an easily recognisable common context with only just over half the respondents stating that they do go well together.

Students' explanations were classified in terms of their clarity and acceptability, or their problematic nature and this for the two judgments relating to whether or not the terms could be grouped. For the two judgments the ratios of clear to problematic are as follows :

"go well"	(a) 28 : 4;	(b) 25 : 3 ;	(c) 14 : 4 ;	(d) 16 : 4;	(e) 16 : 6
"don't go well"	(a) 1: 2;	(b) 2 : 4 ;	(c) 12 : 3,	(d) 3 : 3,	(e) 1 : 4.

It can be seen that on the whole students do not experience too much difficulty in explaining their ideas about *algebraic expression/terms* and *formula/ substitute*, although it is harder to explain why they should not be grouped than why they should. The difficulty increases for the next two sets of words when it comes to putting terms together; in terms of not grouping these same words, students find *multiply - bracket* relatively easy to explain but they have more difficulty with *factorise-product*. The pair *equation - formula* presents the greatest difficulty of all the five pairs as described below. We will now give the most frequent explanation for grouping terms and also examples of problematic explanations.

Algebraic expression - terms: the majority of explanations were in terms of part/whole, that is, "the two words go well together because algebraic expressions are made up of terms". In the problematic responses a number of student-teachers expressed an interesting view of a term when they wrote :"a term is one group of letters or numbers that does not contain an "operator" sign as + - x /".

Formula - substitute : there were essentially two types of explanations first, part/whole explanations for example, "in a formula you substitute values for letters"; second, means/end explanations, for example "you are substituting into a formula in order to obtain a solution" An example of a problematic explanation is: "the expression is made up of letters".

Factorise - product: the category of explanation that predominates is that of the 'opposites sides of the coin' argument, for example, "to factorise you break a product down and to find the product we multiply the 2 factors, so they are the opposite sides of the coin" . Examples of problematic arguments

are :"factorisation of an equation quite often involves factorising into products"; and "product implies multiplication, whereas factorise implies simplification".
For the words ***multiply-bracket*** there are two different sets of reasons to justify: a) that terms could be grouped and b) that they could not. In category a) all students gave as their reason: "when you have a bracket in a statement you have to multiply what is outside by what is inside". In category b) students considered that a bracket can be used for grouping any terms. There were a number of problematic arguments of the kind: for a) "they have a similar operative value"; and for b)"multiplication is an arithmetic operation whereas a bracket is an algebraic operation".

The terms ***equation-formula*** present a situation somewhat different from the four others described above. Whereas with all the other explanations a distinction could be made between problematic explanations and those that were clear and acceptable, such a distinction is harder to make in the present case. Many of the explanations that were clear were also questionable, thus the most frequent and most explicit explanation was of the kind that a formula was really the same as an equation: "Basically they mean the same thing because formulas are equations". There was also a number of other explanations that could be grouped as problematic in as much as it was extremely difficult to interpret the meaning of the written response as in the following examples:"An equation usually consists of a formula"; "both express something in terms of something else". We ourselves think that most of explanations of these two terms, clear or not, were somewhat problematic.

Up to now we have looked at how students imagined explaining in writing an algebraic idea to pupils. In the next question we ask students to criticise three different formulae for the area of a rectangle but this is done indirectly by examining the formulations of three imaginary pupils.

QUESTION 3

The purpose of this question was to examine the criteria students used for the choice of the ***most helpful*** and ***least helpful*** among three explanations given by pupils who were asked to write down a formula for the area of a rectangle of any width and length. The three formulations were as follows: (1) Area = width x length; (2) If **a** stands for the width, and **b** stands for the length, then the formula is : **a x b**; (3) If **A** stands for the area, and **a** stands for the width, and **b** stands for the length, then the formula is: **A = a x b** .

We were able to categorise the students' explanations for their choices into three major categories: (a) Symbolism: the response refers to variables, ways of symbolising quantities, etc. for example, "it does not involve substituting letter for quantities", (b) Pragmatism: the response refers to the efficiency of the formula that is, how easy it is to remember or to work with, for example, "simple, easy to understand, no algebra"; (c) Pedagogic: the reasons have to do with pupils' ability to understand, or to the teaching process, for example, "depends on the level of the pupil, everyone is different".

Most helpful response: Type 1 response was chosen by the majority of students (32 / 40) with about half reasons being pragmatic, that is, the statement: area= width x length, is simpler, easier to remember, or entails no algebra. About a third of the explanations were about symbolism. Examples of reasons in the symbolic category which provide us with students' views of problems associated with symbolism are as follows: "This (type 1) is the best because no "a's" "b's" are introduced which may confuse the child - if they learn it as " A= a x b", what happens when a rectangle has lengths x and y? "; (type1) "because it doesn't involve the substitution of letters for quantities"; (type1)" it doesn't introduce any superfluous terms, tells rather how we calculate area directly. We don't have to relate a number to a letter, rather everything is solid and practical."

2 students chose type 2 and 3 students type 3, and the reasons for their choice were essentially either pedagogic or pragmatic (4 / 5). There were 3 students who made no choice but simply formulated statements of a general pedagogic nature.

The least helpful response: The choice divides roughly equally between type 2 (1 6 / 33) and type 3 (17 / 33), with one pupil choosing type 1 and 6 making no choice but again making statements about the pedagogic or pragmatic nature of the situation. Although symbolic and pragmatic reasons are the two most frequent categories for both choices, pragmatic reasons are more frequent than symbolic reasons for type 2 (9:5) whereas for type 3 they divide equally (7:7). It was noticeable that although both types 2 and 3 were chosen as the least helpful, students did not reject type 2 as an inappropriately formed statement but rather as a rather irrelevant type of formulation in terms of coping with the situation, for instance, "the choice of the letters is irrelevant", "it should choose W, L and A"; "what is the a and the b?"; "what is a x b?".

Again we have chosen a number of reasons from the symbolic categories in order to see how students view the problems associated with this domain: "3 is probably least helpful at this stage because there are a lot of variables included (three) and the use of two different forms of the letter (a, A) could cause confusion." "No.3 is least helpful because it is confusing, it has used abbreviation, i.e. A= a x b".

Conclusions

The results of the analysis of these three questions about pupils' difficulties in algebra gave us through the students' written language some insight into a number of their possible mis-understandings of the terminology, conventions and 'rules' used in algebra. In the first question the concept of ratio revealed itself as particularly difficult to explain, and quite a few students preferred to stay at the level of naming this concept rather than explaining it. Also although the domain of reference in the question is algebra quite a few students referred to the expression "x / x" as a fraction. In the second question the two sets of pairs: factorise - product, and formula - equation gave rise to the greatest number of difficulties in students attempts to explain what they had or did not have in common. More particularly, with the terms formula - equation students were unable to appreciate or express the characteristics of each, and treated them as interchangeable. Lastly the students' choices of the most helpful and least helpful formula for the area of a rectangle chosen from three different formulae revealed that for the most helpful they chose one where the algebraic symbolism was minimal and for the least helpful, they rejected one formula not because it was inappropriately formulated, as was the case, but because it was too complicated. An appropriate formula for the situation was also rejected with only half these students making any reference at all to its symbolism. We intend to follow up this study by looking at experienced teachers and comparing their results with those of the PGCE students, as well as, at the later date, comparing the results of both groups of teachers with the results of the pupils' study from which this research originated.

References:

Arnold, H (1973), Why children talk: Language in the primary classroom. *Education in Teaching*, Vol. 91, pp.25-29

Howson A.G. (1980), Language and the teaching of mathematics. *Proceedings of the Fourth International Conference of Mathematics Education*, Birkhause, pp.568-572

Austin, J.L and Howson, A.G. (1979), Language and mathematical education. *Educational Studies in Mathematics*, Vol. 1, No.2 pp.161 - 197.

ALGEBRA - CHOICES IN CURRICULUM DESIGN

An Exploratory Teaching Experiment

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It is generally accepted that students' failures in algebra are extensive, and of two kinds. First, there are characteristic errors in manipulation, such as $4(n + 5) = 4n + 5$, or $(x + 8)/(x + 2) = 8/2$, or $x - 5 = 7$ giving $x = 2$; many of these relate to incorrect responses to perceptual cues in the expressions (Saad, 1960; Carry, Lewis and Bernard, 1980). Secondly, there are more global conceptual breakdowns, such as the failure to appreciate the significance of checking the solution of an equation, thus regarding the performance of the solution process as the aim of the task, rather than the obtaining of a value of x which makes the equation true (Lee and Wheeler, 1987). Some research attention has also been focused on the modes of interpretation of the letter symbol - as object, evaluated as a number, as specific unknown, generalised number or variable (Kuchemann, 1981).

Further uses of letters, not included in Kuchemann's study are, for example, those shown in the expression $y = ax + b$, which denotes a function, connecting the variables x and y, whose properties are those of linearity, with scale factor a, and with $y = b$ when $x = 0$. The a and b here are generally called 'parameters'; they are generalised numbers, but play a different role from the x and y, which are in a sense 'dummy' variables, serving merely to enable the function to be expressed. There are also uses of letters in geometry; $y = ax + b$ could itself describe a line, (of gradient a and intercept b) in coordinate geometry. And in the geometry of transformations, considering the symmetries of a rectangle we may write $r = hv$, where the letters denote particular transformations, (analogous to particular numbers). But unknowns and generalised numbers can also appear, as in "$vx = hrhv$, what is x?" and "$xy = yx$, where x, y are any of r, h, v".

The question arises whether the algebra curriculum should be structured according to types of manipulation, as is traditional, or according to modes of use of the letter, or by modes of algebraic activity, such as generalising, equation-forming and solving, and so on.

One recently published course (to which Kuchemann and Harper have contributed) adopts the second of these following uses of the letter. In this, the algebraic content for the first two years (ages 11 and 12) begins simply by getting the students to use a letter for a temporarily unknown number, which is then immediately to be calculated. Generalised numbers appear in the form of simple expressions for drawing zigzags, spirals and sets of rectangles eg, sets of rectangles with length and breadth 2k, k + 2; values are to be given to k and the set of rectangles drawn. Another set includes some of type 2k, k + k which are to be picked out as forming squares (NMP, 1987).

Our view is that the curriculum should be designed to ensure that the whole range of uses of letters (and indeed of other symbolisms too) should be covered, but that the structuring should be on the basis of types of situation which give rise to distinct modes of use of algebra. Just as in language learning, separate attention may be devoted to descriptive, expressive and persuasive forms of writing (and others), so in mathematics there is a whole complex of procedures appropriate to different types of algebraic situation. We distinguish in this way:

1) Generalising (mainly in number situations).
2) Situations leading to forming, solving and interpreting equations.
3) Functions and formulae.

As well as these types of situation in which algebra arises, we need to focus attention at appropriate points on

4) General number properties, including their manifestation as rules for manipulating expressions.

The mode of learning activity is an important feature of this course; it is to explore a chosen situation containing a number of variations, and to provide for the making up, solving and checking of problems by the students within this framework.

SITUATIONS USED

We shall describe the use of Generalising in Number Situations with an able group of 14year olds; and of Forming and Solving Equations with a middling but above average group, also aged 14.

GENERALISING IN NUMBER SITUATIONS

This material consisted of two situations, Consecutive Numbers and Patterns in the Number Square. The first involved recognising that the sum of three consecutive numbers is divisible by 3, and extending this to 4, 5, 6, ... numbers, eventually generalising to any number of consecutive numbers. The method of denoting the numbers by n, n + 1, etc was introduced by the teacher, and readily picked up and used. However, the resulting expressions 3n + 3, 4n + 6, 5n + 10, etc were generally not read as implying divisibility by 3, by 2 but not 4, by 5, etc, but regarded as the end point of the task themselves, or else used to spot further patterns by differencing.

$$\begin{array}{ccccc} 3 & & 6 & & 10 \quad \ldots \\ & 3 & & 4 & \end{array}$$

The distributive law 3n + 3 = 3(n + 1) was shown, and accepted, but not much used by the pupils.

In the second situation, a 3 x 3 box was chosen in the number square, and the opposite corner numbers

$$\begin{array}{ccc} 17 & 18 & 19 \\ 27 & 28 & 29 \\ 37 & 38 & 39 \end{array}$$

added to give the same sum: 17 + 39 = 19 + 37; this was expressed $x + (x + 22) = 2x + 22$ compared with $(x + 2) + (x + 20) = 2x + 22$.

The class were asked, working in groups of three, to find other such patterns in this square, and in the addition and multiplication squares.

A variety of these were found, and the significance of the use of the x as guaranteeing the generality of the result, across all possible positions of the box within the square, was generally appreciated. However, none of the class ventured into the more difficult multiplication square. In an operational situation, as distinct from a short exploratory experiment, it would be important to include this and other more varied cases, as the assignment of x (and y) to denote the number in a general cell takes different forms, and the development of this experience is an important teaching objective.

FORMING AND SOLVING EQUATIONS

This began with the following problems.

1. There are two piles of stones. The second has 19 more stones than the first. There are 133 stones altogether. Find the number in each pile.
2. This time the first pile has seven times as many stones as the second; there are 40 altogether.
3. 3 piles; the first has 5 less than the third, and the second has 15 more than the third. There are 31 altogether.

Students were asked to solve the first two problems, no method being specified. They did so, by trial, about half getting the correct answer to the first, the remainder halving the 133 before subtracting 19. They were then shown how to solve the same problem using algebra. The number in the first pile was denoted by x and the equation $x + x + 19 = 133$ formed and solved to give $x = 57$, and the numbers in the two piles 57 and 76, which were checked to add to 133. They were then asked to solve the same problem, but using x for the second pile; thus obtaining a different equation, the solution $x = 76$, but the same numbers for the two piles.

Following this they were asked to work, in groups of three, at solving the third question, taking in turn each of the three piles as x; and to compare their results. On the following day each group was asked to make up and solve 3 similar problems, two easy and one hard, to be attempted by another group.

This led to a lot of insight into the way different x-assignments affected the expressions, turning + into - and multiples into fractions. It also led to an unexpected degree of richness in the problem statements. As well as four bean bags and the number of pupils in three rival schools we had

> "A nuclear scientist must complete 4 experiments to save the world, and he has 23 days to do them in. The first will take twice as long as the second ..."

(I had originally wished to replace 'piles of stones' by a more exciting basic situation, but had been unable to think of a similarly flexible one; I needn't have worried).

CONCLUSIONS

In this last situation, the letters introduced were clearly 'specific unknowns'. But, as in Generalising, the main difficulties lay in expressing relations such as 'pile 3 has 15 more stones than pile 2', when pile 2 was x, making pile 3 'x + 15'; and more so when pile 3 was x, needing a reversal to make pile 2 'x - 15'. '10 less than x + 15' was another step up in difficulty.

However, although this was observed as a serious obstacle for some students in the early lessons, on being offered the answer, they soon picked up how to do it, and in the school examination question on this work, no student failed to formulate an equation, though there were some 'reversal' mistakes. It is hard to see any psychological distinction between "if this unknown number is x, what is 15 more than x" and "x may be any one of a whole possible set of numbers; but y is always 15 more than x; how do we write it?".

The most significant difficulty in this equation work concerned the manipulation of the expressions obtained. For example, three piles correctly obtained as x, x and x x 2 were added to give 3x x 2; even x, x and 2x became 3(2x) in one case. This is not an error at the strategic level, but a technical one of not knowing how to combine terms. If x + 15 + 2x can be combined to give 3x + 15, why is the above collection wrong? This involves considering how + and x operations commute and associate.

Thus, with these classes, the distinction between x as a specific unknown and x as a generalised number did not appear to relate to conceptual difficulty. More important, it seemed, was the fact that in both types of situtions, Generalising and Equation-forming, what was being denoted by the x was a clearly recognised, almost concrete element of the well understood situation - either the number of stones in that pile, or the corner (or middle) number in one of those boxes on the number square "like the 17 in that one".

It is also clear from experience with these situations that there are 'scripts' for dealing with generalising and with equation-use, and these concen not simply the use of the x, but also the whole procedure, and the modes of reasoning which are appropriate. In the one case it is forming, solving, checking solution, and in the other assigning x and forming and transforming two expressions until they are seen to be the same.

OTHER ASPECTS

Other modes of algebraic activity explored in this experiment included developing one's skills at manipulating equations, and relating functions and formulae, in particular 'reading' formulae to recognise the functions embodied in them (such as in A = L x B, the relation between L and B with A held constant). Details of these, and discusion of pre-post test results, may be found in the full report.

REFERENCES

Bell, A, Malone, J and Taylor, P (1987): Algebra: an exploratory teaching experiment. Curtin University, Perth and Shell Centre, Nottingham.

Carry, L, Lewis, R and Bernard, J (1980): A Psychological Study of Equation Solving. University of Austin, Texas.

Kuchemann, D E (1981): Algebra in Hart, K: Childrens Understanding of Mathematics, 11-16. Murray.

Lee, L and Wheeler, D (1987): Algebraic Thinking in High School Students; Their Conceptions of Generalisation and Justification. Concordia University, Montreal.

NMP (1987): National Mathematics Project. Longman.

Saad, G (1960): Understanding in Mathematics. Oliver and Boyd.

NUMBER NAMING GRAMMARS AND THE CONCEPT OF "10"

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This research investigated the influence of the structure of number naming systems on children's numerical cognition. Four monolingual English-speaking (ME) children (mean age 5.8) and four bilingual Vietnamese/English-speaking (BVE) children (mean age 6.2) in the same year 1 (first year of formal schooling) class were given 15 weekly individual teaching sessions in numeration and simple numerical operations. Teaching sessions were videotaped, and subsequent analyses used the children's responses to chart the development of their conceptual structures for "10". It was hypothesised that the BVE children would evidence both different final structures and different rates of development from their ME comparators. Significant differences both between children and between groups were discerned, some children clearly demonstrating an ability to reflect upon the grammatical structure of the Standard Number Word Sequence.

A substantial amount of anecdotal evidence from the Australian teaching profession points to superior performance on numerical tasks of students of Oriental descent, and there is strong formal evidence of the same phenomenon from international settings (Husen, 1967; McKnight et.al., 1987). Some of the things that these students have in common include high motivation levels, traditionally structured family backgrounds and a number naming system which may be described as transparent.

SOME TRANSPARENT NUMBER NAMING SYSTEMS

	VIETNAMESE	MANDARIN	JAPANESE
1	mot	yi	ichi
2	hai	er	ni
3	ba	san	san
4	bon	si	shi
5	nam	wu	go
6	sau	liu	roku
7	bay	qi	shichi
8	tam	ba	hachi
9	chin	jui	ku

10	muoi	shi	ju
11	muoi mot	shi yi	juichi
12	muoi hai	shi er	juni
13	muoi ba	shi san	jusan
14	muoi bon	shi si	jushi
15	muoi nam	shi wu	jugo
16	muoi sau	shi liu	juroku
23	hai muoi ba	er shi san	niju san
37	ba muoi bay	san shi qi	sanju shichi

The names given to the concept "13" -- "muoi ba", "shi san" and "jusan" -- contain semantic cues which reflect the decimal structure of the modern counting system. Logic and conventional wisdom about how children learn seem to suggest that such a number naming system may advantage young children who use it in operating on numbers. From an information processing perspective for example, such a system uses fewer "bits" than, say, the Germanic or Romance languages, so some degree of cognitive economy might be operative.

There is clear evidence of these cultural differences in tasks of numerical competence both at higher (Lesser, Fifer and Clark,1965) and lower (Lui Fan,1981) age levels. In some studies (Ginsburg, 1981; Posner, 1982; Hatano, 1982; Lancy, 1983), this differential in numerical performance has been directly linked with language structures, while in others (Ayabe and Santo, 1972; Kitano, 1974;) an attitudinal explanation has been offered. Although both Chinese and Japanese subjects have consistently shown significantly higher competence on numerical tasks (Husen, 1967; Lui Fan, 1981; Stevenson et.al., 1982,1985; Mcknight et.al., 1987)), and these cultures both use a transparent number naming system, it may be that this artefact can be explained in terms of a generally different cognitive (Tsang, 1984) or perceptual (Hsi and Lim, 1977) framework, or even a fundamentally different school curriculum (Easley, 1983).

Certainly, the link between Standard Number Word Sequence structure and children's early number abilities has not been adequately investigated. Recent commentary on SNWS development for example, places little significance on the relationship between the grammatical structure of that sequence and children's mental representation of number.

> "..... much of the development of the young child's numerical abilities involves the rote learning of the first 12 or 13 number words and the generative rules for producing the subsequent number words." (Gelman and Gallistel, 1978:79)

> None of these authors found evidence that children understood the teens structure of the number words at the time they were acquiring those words. The number word sequence thus seems to be for young children an unstructured list until twenty (or perhaps twenty nine), and then the decade structure is evident." (Fuson et.al., 1983:54)
>
> "...the English SNWS segment that contains all whole number words from one to twelve must be fixed in the child's memory. After thirteen a preliminary composition procedure may set in, and after twenty a general one." (Steffe, von Glasersfeld, Richards and Cobb, 1983:26)

The questions of whether children can become aware of the significance of the structural aspects of spoken language by reflection, or whether language can itself structure subconscious thought are still unanswered. There is evidence, although not from the mathematical register, and more often from adult subjects, that both phenomena occur. Meyer and Schvaneveldt (1976) found consistently that a test word which is immediately preceded by a semantically related word ("green ..... grass") will be more quickly recognised than an unprimed test word. Fischler (1977) argued that semantic relatedness is the main determinant of facilitation in lexical decisions. Both results seem to indicate attention to components of the verbal stimulus. Taken further, these results could perhaps be used to infer that a Vietnamese child who could quickly recall "ba + bon = bay" (3 + 4 = 7) would employ similar structures to recall "muoi ba + bon = muoi bay" (13 + 4 = 17).

The central aim of this study then, was to examine the influence of number naming grammars on children's numerical development. In particular, it analysed the conceptualisation of "10" in young children from two linguistic backgrounds which employ either a transparent (Vietnamese) or opaque (English) number naming system. In doing so, it drew on an experimental and theoretical framework recently developed by Steffe and his co-workers at the University of Georgia.

A teaching experiment was undertaken with 5-6 year-old English-speaking and Vietnamese-speaking children in the period July-November 1985. Initial individual interviews were conducted with 18 children from the same year 1 (the first year of schooling) class. Tasks were derived from Steffe's (1983) task sequence, and oriented towards assessing facility with the number word sequence, both in English and the first language. Following these initial interviews, 4 monolingual (English) (ME), and 4 bilingual (English and Vietnamese)(BVE) children were selected from the same class on the basis of broadly comparable performance on the standard number word sequence in their first language. These children received individual instruction for 15-20 minutes each week for 15 weeks, spread over the period July-November 1985.

Sessions were videotaped for subsequent analysis, and the overall orientation of the instructional sessions was towards game activities designed to lead the child towards the adult model of mathematical competence through the fulfilment of the following objectives indicative of the competencies required in primary school mathematics:

1. --to foster competence with the SNWS in English and the first language;
2. -- to build the conception of the number system as concatenations of ten.
3. -- to develop the coordination of this sequence with presentations of countable unit items;
4. -- to consolidate the operational schemes of each child;

Case study analyses focussed on the development of each child's meaning structures for "10", and these structures were inferred from the video record of each child's actions (or intentions to act) in problem situations. The predicted behavioural manifestations of the child's conceptual structure for "10", derived from Steffe and von Glasersfeld (1983) were as follows:

CONCEPTUAL MANIFESTATIONS OF "10"

1. **A SPECIFIED PERCEPTUAL COLLECTION**.(SPC) This refers to any counted collection of 10 perceptual items that cannot be re-presented.
2. **A PERCEPTUAL UNIT**.(PU) Perceptual collections of 10, which are only temporarily established in the visual field of the child, and which derive salience only from the items of the collection and the patterns in which they are arranged, are abstracted through the ability to re-present counting activity, which leads to the realisation that any two perceptual collections of 10 will have a common feature -- if they were counted, "10" would be the result.
3. **A COUNTABLE PERCEPTUAL UNIT**.(CPU) If a child can coordinate the Decade Number Word Sequence with specific perceptual units which each contain 10 items, it is said that, for the child, these units are countable.
4. **A FIGURAL PATTERN**.(CFU) This is a pattern of 10 counted items that can be re-presented. It may take the form of 2 open hands, or a figural image of a bundle or base 10 long, or may have idiosyncratic significance. If it is coordinated with the DNWS it is called a countable figural unit.
5. **A COUNTABLE MOTOR UNIT**.(CMU) This type occurs in the context of counting perceptual units of 10 using the DNWS, when a motor act like putting up a finger or pointing with a finger is used as a substitute for perceptual units of 10 that are screened from sight.
6. **A NUMBER WORD PATTERN**.(NWP) If children are capable of re-presenting and reviewing the results of a counting activity and pulling from it the recurrent results of making intuitive extensions of 10, they are sometimes able to construct a number word pattern from a point within a decade, "5, 15, 25,....", incrementing by implied 10 counts.

7. **A NUMERICAL COMPOSITE UNIT.(NC)** This conceptual re-organisation enables the child to take a perceptual collection of 10 items as 1 unit, while maintaining its numerosity. This unit refers to any pattern of 10 that is the result of an integration, the focus being on the elements and the pattern not being taken as 1 thing.

8. **AN ABSTRACT COMPOSITE UNIT.(AC)** The ability to coordinate counting by tens and ones when counting on, is symptomatic of the use of ten as an abstract unit.

RESULTS AND DISCUSSION

The research hypotheses for the children in the study were:

1. That the BVE children (Hao, Qyen, Ai, and Tony in order of age) would exhibit some of the described conceptual manifestations of 10 earlier than the ME children (Kim, Peter, Kylie, Sean).
2. That the ME children would not exhibit any of the described conceptual manifestations of 10 earlier than the BVE children.
3. That the BVE children would exhibit conceptual manifestations of 10 not exhibited by the ME children.
4. That the ME children would not exhibit conceptual manifestations of 10 not also exhibited by BVE children.

HYPOTHESIS 1 was accepted. BVE children Tony, Ai and Hao all evidenced 10 as an Abstract Composite earlier than ME children, and apart from one exception, the clear tendency was for BVE children to produce 10 as a Perceptual Unit, 10 as a Countable Perceptual Unit, 10 as a Countable Figural Unit, 10 as a Countable Motor Unit, and 10 as a Number Word Pattern earlier than the ME children.

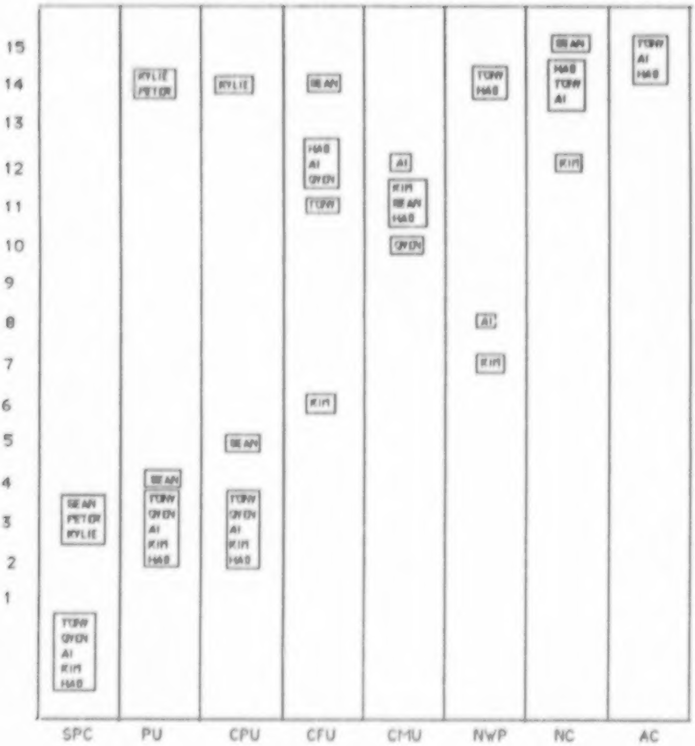
The notable exception was Kim, an ME child who attained 10 as a Countable Figural Unit, 10 as a Number Word Pattern and 10 as a Numerical Composite before any BVE child. She was the youngest of all the subjects, yet she entered the study with competencies clearly in advance of her comparators. Her Sequencing by 1 and Decoding routines were observed to be operative from July, and her Sequencing by 10 routine attained an operative level in session 9, before any of the other children.

Kim's exceptional performance can be explained in terms of the fact that she abstracted number before six of the other children. This, coupled with her early mastery of the English homonymic transformations, enabled her to focus on the components of a number name, make integrations on each component, and anticipate the results of extending or contracting each component.

HYPOTHESIS 2 was rejected. Kim's early use of 10 as a Countable Figural Unit, 10 as a Number Word Pattern and 10 as a Numerical Composite clearly showed that it was possible for an ME child to attain these concepts before BVE children. Yet, it has already been noted in the discussion of HYPOTHESIS 1 that

STRUCTURES FOR 10

(Showing the teaching session at
which each structure was observed
for each child)



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if Kim's performance is put aside, there was a clear tendency for BVE children to attain more complex structures earlier. Certainly the structures attained by Kylie, Sean and Peter (ME children) were slower to develop than those of the other children, Peter for example never abstracting 10 as a Countable Perceptual Unit, and Kylie only attaining it at the last session.

HYPOTHESIS 3 was accepted. Tony, Ai and Hao (BVE children) all exhibited 10 as an Abstract Composite -- a structure not exhibited by any ME child.

HYPOTHESIS 4 was accepted. There were no identifiable structures exhibited exclusively by ME children.

GENERAL RESULTS

The central aim of this study was to investigate whether children whose first language employs a transparent number naming system develop conceptual structures for "10" which are demonstably different from those of children from an English-speaking background. Demonstrable differences in both conceptual structure for 10 and rate of development of those structures were found. Additionally, there were differences in the rate of development of the Sequencing by 10 routine. These differences were evident both between children and between BVE and ME groups.

CLASSROOM IMPLICATIONS

This study employed a case study approach, so it would be hazardous to attempt to extend these results beyond the limits of sampling. It has however shown that some children appear to be capable of reflecting upon the structure of verbal utterances and abstracting concepts of numeration as well as techniques of addition from the information contained in those utterances. In addition it showed that the structure of the English SNWS with its extensive reliance on homonymic transformations effectively inhibits the attempts of some children to abstract numeration conventions from the number naming grammar.

This suggests that it may be appropriate to introduce young European children to alternative transparent Number Word Sequences in order to focus their attention on the isomorphism between the verbal and numeric representations for number. Such an exposure (say, to Chinese, Japanese, Vietnamese, Esperanto, or some other invented system) could be justified not only from a multicultural, but also from a cognitive standpoint.

Future research should explore the usefulness of employing a transparent number naming system to remediate numeration misconceptions and algorithmic errors in older children.

REFERENCES

Ayabe,H., and Santo,S., (1972) Conceptual tempo and the oriental American. The Journal of Psychology, 81, 121-123.

Easley,J., (1983) A Japanese approach to mathematics. For the Learning of Mathematics, 3(3), 8-14.

Fischler, I., (1977) Semantic facilitation without association in a lexical decision task. Memory and Cognition, 5, 335-339.

Gelman,R., and Gallistel,R., (1978) The Child's Understanding of Number. Cambridge,Mass.: Harvard University Press.

Ginsburg,H., (1981) The development of mental addition as a function of schooling and culture. Journal of Cross-Cultural Psychology,12(2), 163-178.

Hatano,G., (1982) Learning to add and subtract: A Japanese perspective. In Carpenter,T., (et.al.) Addition and Subtraction: A Cognitive Perspective. New York: Erlbaum.

Hsi,V., and Lim, V., (1977) A Summary of Selected Research Studies on Cognitive and Perceptual Variables. Berkeley,Ca., Asian American Bilingual Center.

Husen,T.,(1967) International Study of Achievement in Mathematics: A Comparison of Twelve Countries. New York: Wiley.

Kitano,H.,(1974) Race Relations. New Jersey: Prentice-Hall.

Lancy,D., (1983) Crosscultural Studies in Cognition and Mathematics. New York, Academic.

Lesser,G., Fifer,G., and Clark,D., (1965) Mental abilities of children from different social classes and cultural groups. Monographs of the Society for Research in Child Development, 30,4.

Liu Fan, et.al., (1981) A study on the development of number concepts and operational abilities of 7-12 year olds in 10 districts in China. Acta Psychologica Sinica., 13(2), 135-149.

McKnight, C., (1987) The Underachieving Curriculum. Champaign: Stipes.

Meyer,D., and Schvaneveldt, R., (1976) Meaning, memory and mental processes. Science, 192, 27-33.

Posner,J., (1982) The development of methematical knowledge in two West African societies. Child Development, 53, 200-208.

Steffe,L., von Glasersfeld,E., Richards,J., and Cobb,P., (1983) Children's Counting Types: Philosophy,Theory and Application. New York: Praeger.

Steffe,L., and von Glasersfeld,E., (1983) The Construction of Arithmetical Units. Proceedings of the Fifth Annual Conference of the IGPME.(Nth. America) 293-303.

Stevenson,H., Stigler,J., Lucker,G., Lee,S., Hsu,C., and Kitamura, S., (1982) Reading disabilities: The case of Chinese, Japanese and English. Child Development,33, 1164-1181.

Stevenson,H., Stigler,J., Lee,S., and Lucker,G., (1985) Cognitive performance and academic achievement of Japanese, Chinese and American children. Child Development, 56, 718-734.

Tsang,S., (1984) The mathematics education of Asian-Americans. Journal for Research in Mathematics Education, 15(3).

THE KINDERGARTNERS' UNDERSTANDING OF DISCRETE QUANTITY

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Abstract

When the natural numbers are viewed as the means to measure discrete quantities, the notion of quantity can then be considered as a pre-concept of number. This paper reports the results of a study regarding the kindergartners' understanding of discrete quantity. Our investigation shows that three distinct components of understanding can be found among this age group. All 24 children tested indicated they had an intuitive understanding as evidenced by their ability to estimate quantity on the basis of visual perception. A more advanced level of comprehension , that of procedural understanding, was evidenced when each child proved able to use procedures based on one-to-one correspondences to constructs sets that were larger, smaller, or equal than a given one. A third component of understanding, that of abstraction, was studied through various tasks ascertaining the subjects' ability to perceive the invariance of quantity with respect to various surface transformations, that is, changes in the disposition of the objects which did not affect the given quantity.

If one views the natural numbers teleologically, that is in terms of their utilization, one must take into account both their **cardinal** function, which enables us to measure the quantity of objects included in a discrete set (Vergnaud,1979) and their **ordinal** function which enables us to determine the rank of an object in an ordered set. Thus, it can be seen that the concepts of **quantity** and **rank** are in a sense fundamental schemas on which the notion of number can be built. This distinction leads to a finer discrimination between number, which is a mathematical construct, and quantity and rank, which are rather physical constructs. The children's understanding of these physical quantities has often been confused with their understanding of number as for example in Piaget's classical experiment on the conservation of "number" (Piaget & Szeminska,1941/1967). In fact, since in his test, subjects are not required to enumerate any of the rows of objects, the notion of number can hardly be invoked and thus the task should be considered as pertaining to the conservation of quantity.

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This past year we have investigated the kindergartners' understanding of discrete quantity and of the notion of rank. We report preliminary results of this ongoing study in two companion papers. The present one deals with quantity.

In our analysis of the notion of discrete quantity, we have postulated three distinct components of the child's understanding of this conceptual schema. A first component which can be viewed as an **intuitive understanding** of this concept reflects a type of thinking based essentially on visual perception. At this level, children can easily compare two sets and estimate visually who has **more** and who has **less**, or if one has **as many as** the other; in comparing two sets they often can determine by mere perception which one has **many** and which one has **few** or **little**.

A more advanced level of operation is involved when children can actually use a rational procedure enabling them to make these judgments about quantity with reliability and precision. The acquisition of such means can be perceived as bringing about a higher level of comprehension which we describe **as procedural understanding** . The notions of 'more', 'less' and 'as many' can all be assessed by using procedures based on **one-to-one correspondences**. While still being non-numerical in the sense that no enumeration is involved, such procedures can be carried out physically by the children and provide them with an assurance which they cannot obtain from mere visual estimation.

Still a more advanced level of comprehension is evidenced when the child's conception of quantity becomes more stable and can resist various **surface transformations**. Piaget's conservation of quantity is one such example. Young children believe that after one of two equal rows of objects has been elongated, it somehow must contain more than the other row. There are many other such transformations which can test whether or not the concept of quantity is firm enough in the subjects' mind to overcome the erroneous information they obtain from their visual perception. This detachment from visual perception leads them to a level of understanding which we call **abstraction**. Of course, it does not occur all at once, and in fact one can usually establish a hierarchy among various tests assessing the **invariance** of a given concept with respect to specific transformations.

The present paper describes the many different tasks we have designed to assess the kindergartners' knowledge of quantity, as well as the exact wording of the questions. These tasks have been used in semi-standardized interviews with 24 children (average age 5:8) coming from three different schools in Greater Montreal. The interviews required one session of about 30 minutes with each child. Each interview was videotaped .

Intuitive understanding

The intuitive understanding of quantity was assessed through two tasks based on visual estimation. In the first task, the interviewer laid out 25 identical cubes in front of the subject and 7 cubes in front of herself while asking:

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Here are some cubes for you (25) **and here are some for me** (7).
(a) **Just by looking at them, can you tell me who has more, you or me ?**
(b) **Can you tell me who has less, you or me?**
(c) **Can you tell me where there are many?**
(d) **Can you tell me where there are few** (if needed,**not many**)?

All the children interviewed answered these questions without hesitation thus indicating that they could handle these visual estimations with ease. Only two children did not understand the word "few" but they grasped the meaning of "not many". Hence even among the youngest children in kindergarten, between the ages af 5 and 5-l/2, these subconcepts and the vocabulary associated with quantity have been acquired.

The second task was aimed at testing the child's visual estimation in the context of equal sets. Two sets of identical cubes were laid out randomly, one set in front of the child, and one set in front of the interviewer:
(a) **Here is set for you and here is a set for me. Just by looking at them, can you tell me if you have as many as I have?** (if needed, **the same as I have**)
(b) **What would you do to make sure?** (and if the child suggests counting)
(c) **Do you have another way to make sure, without counting?**

Among the 11 younger children in our sample (aged from 5:3 to 5:8) 5 of them could not estimate visually if the two sets were equal. Among the 13 older children (aged from 5:9 to 6:2), there were 4 such subjets. There seems to be some difficulty at the level of vocabulary. In French, the words we use are "En as-tu autant que moi". Several children do not understand this and we then resort to other expressions such as "En as-tu pareil que moi?" or "En as-tu la même chose que moi?"

The additional questions have proved informative. All except five children suggested they could make sure that the two sets were equal by counting. This is clear evidence indicating that the numbers learned by these children have acquired a very strong cardinal meaning. Only a few of the children could think of other means of comparison. A couple of children suggested putting the sets in a one-to-one correspondence and a few others made a rectangular array of 2x4 with each set of cubes and then indicated that they were the same shape.

Procedural understanding

The tasks were designed to assess if the children could use a one-to-one correspondence in comparing two quantities. The interviewer aligned eight green cubes in front of the child and gave him or her 10 red cubes while raising the following questions:

Here is a row of green cubes, and I'm giving you these red cubes.
(a) **Can you make me another row just like mine?** The child will usually lay out the 10 cubes. **Do you think that we have as many cubes in one row**

as in the other? How can we make them the same?
(After each task, the child's row is removed)

(b) **Would you now like to make a row with more cubes than in this one?**

(c) **Can you make a row where there are less cubes than in this one?**

(d) **Can you make a row where there is one more cube than in this one.**

These tasks were aimed not merely at verifying if the child could recognize these quantified relationships but if the subject could actually use a one-to-one correspondence to generate these required sets. Using the interviewer's row as a template, all 24 subjects were able to generate an equal set. Only one child failed to construct a row with more cubes, while two subjects failed to produce one with fewer elements. Surprisingly, it is the last question which seemed to cause some difficulty. Six of the 24 children (three in each age group) were unable to generate a set which contained one more cube than the given row.

Abstraction

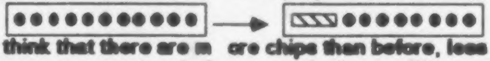
As mentioned in the introductory remarks, abstraction refers to that level of understanding associated with the construction of invariants. Four distinct tasks have been designed to test the child's perception of the invariance of quantity: with respect to the visual perception of the elements, with respect to their random configuration, with respect to the elongation of a row, and the classical Piagetian conservation task. In the words of Freudenthal (1983,p.84), these tasks are aimed at testing the invariance of a set of discrete objects under a change of perspective and under "shake" transformations, that is, resulting from a changed disposition of the set.

Invariance with respect to the visual perception of the elements

Prior research (Anne Bergeron, N.Herscovics, J.C.Bergeron,1986; L.Steffe, E.von Glasersfeld, J.Richards, P.Cobb,1983) has shown that children experience great difficulties in counting partially hidden sets. Steffe et al.have coined the expression **"perceptual units"** to describe the situation when distinct objects are perceived by the senses, be they visual, auditory, or motor. Research reported by A.Bergeron et al. indicates that the visibility of all the objects in a row of chips can be a crucial factor in the child's perception of cardinality. In fact, they reported that with six chips covered in front of them in a row of 11 , some children only counted the visible ones when asked **how many chips altogether were glued on the large cardboard?** Most of those who were asked to count on from 6, continued to count on to eleven but were unable to say how many chips were on the cardboard they had just enumerated. The most common procedure used by children was **figural counting** (Steffe et al.1983) which refers to the child's enumeration of **imagined** units while pointing with a finger over the hidden part of the row. In most cases, subjects using this strategy failed to arrive at a correct count although they all knew the number of chips being hidden. Clearly, the perception of units was of prime importance in the child's construction of numerical units. Thus the investigation of perceptual units with regards to quantity, in a non-numerical context, was of great interest.

The first task involved a row of 11 chips glued on a cardboard. The child was told:

Here is a large white cardboard with chips glued to it. Look,I'm covering a part of it.

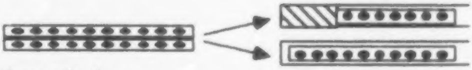


Now, do you think that there are more chips than before, less chips than before, or the same amount as before, on the large white cardboard?.

In case the child might construe that the three hidden chips should not be taken in consideration because they were at one extremity, this was followed up with a second task in which three middle chips were hidden. And since the child might not understand that the whole cardboard was involved, a third type of situation was presented. The interviewer aligned 10 chips in front of the child and provided him or her with another set while asking for another row right next to hers. The subject was then asked if the two sets were the same. The 3 chips of the interviewer's row were covered by a small cardboard and the question was raised whether or not the two rows still had the same amount of chips.

The results are quite striking. Only one single child thought that the row with the cardboard hiding three chips had the same amount as before in the first two tasks, and that the two final rows in the third task were equal. Every other child stated that the quantity had changed. Thus, the three tasks produced an extremely consistent behaviour. Hiding a part of the set in front of the child removed the hidden objects from their consideration. Many subjects stated this quite openly: "When you hide the chips, it is as if they are not there anymore". It might be argued that this kind of response is induced. Perhaps the child associates hiding with some kind of game and ignoring the hidden part is some kind of make believe behavior. In view of this conjecture, we repeated these three tasks in even simpler situations. Of course, these new tasks were not carried out at the same time as the first ones, but during the second interview.

The child was presented with two cardboards on which 10 chips were glued. They were placed next to each other so that the subject could establish through an obvious one to one correspondence that the same quantity was involved. Both the child and the interviewer put one such cardboard in a plactic bag, the child's bag being completely transparent and the interviewer's being partially opaque so as to hide three chips:



The subject was then asked:

Now in your bag, do you have the same amount of chips as I have in my bag?

For the other two tasks involving a single cardboard, plastic bags were provided, the opaque part hiding respectively the last three chips and the middle three chips in the row.

In these last three tasks, the wording of the questions was such that there could hardly be any misunderstanding. The "chips in the bag" could not be taken as meaning the "visible chips". The hiding now was no longer active but incidental. Thus no hiding game could be inferred. Yet, the responses of the 24 subjects changed but little. Only two more children stated that the quantity was the same in the bag. These results indicate the extent to which the child depends on visual thinking. Even when touching the hidden objects in the plastic bags, our subjects stated without any hesitation that there were fewer objects present in them.

Invariance with respect to random configuration.

Another set of tasks dealt with the invariance of the quantity of objects with respect to their random configuration. A set of 9 cubes was disposed randomly before the child. The interviewer then very carefully, using one finger to spread out the set, displaced one cube at a time, making sure that the objects were at all times visible to the child. Prior work had shown that it was important to perform this slowly and with care for otherwise the subject might believe that through some sleight of hand, some cubes had been removed or added.The child was then told:

Here are some cubes. Take a good look. I am going to move them around. If I put them like this (spreading them out) **can you tell me if there are now more cubes, or less cubes, or the same amount of cubes as before?**

The same pattern of questioning was used later on in the interview to verify the effect of **contraction** of a set of randomly displayed cubes.

Comparing the responses to both the dispersion and the contraction tasks provides an indication of the stability of the child's perception of this particular invariance. Of the 24 subjects in our sample, 13 thought that the quantity had not changed in either task, while 6 pupils believed that the quantity had changed in both cases. These two numbers add up to 19 which represents an index of stability of 79%. The remaining 6 children (three from each age group), that is 21% of our sample, must be considered as transitional, since their answers varied in both tasks, three of them believing that the quantity had changed in the expansion but not in the contraction. These results seem to indicate that the surface covered by the cubes, that is, the space they occupy, might be a determining factor for those children who thought that the quantity had changed.

This conjecture could easily be verified with the next two tasks. The first one had the 8 cubes in a paper plate. The 8 cubes were then moved around in the plate.Since the space occupied did not change, this would verify if the mere act of moving the cubes would have any effect. We also provided an additional task in which the plate of cubes was simply **rotated** in front of the subject.

Results were rather surprising. Questioning the children who thought that quantity had changed in the dispersion and contraction tasks revealed that nearly all of them (five out of six) thought that the quantity had changed when the cubes where moved around in the paper plate. This indicates that the mere motion of the objects without any change in the space occupied can affect the child's perception of quantity. The rotation of the plate proved somewhat more successful since four of these six children stated that it did not affect the quantity. The responses of the transitional group proved to be mixed. Two of the five children thought that quantity was affected by both the motion within the plate and the rotation of the plate, two children stated that quantity was not affected in either case, while one child believed that it was affected in the first case and not in the latter one. As expected, all the children questioned from among those 13 who did not think that quantity had varied with dispersion or contraction responded that it did not change in the last two tasks.

Invariance with respect to the elongation of a row

The last two tasks on the invariance of quantity dealt with the visual impact of the elongation of a row. The first of these involved 11 cubes which were aligned in a row in front of the child following which, the cubes were then spaced out evenly resulting in an elongated row. Children were told:

Here is a row of cubes. Look, I'm going to spread them out. Now do you think that there are more cubes,less cubes,or the same as before?

The next task was the classical Piagetian test on the conservation of quantity. The interviewer laid out a row of 7 cubes and handed out 10 other cubes to the child. The excess cubes were removed while the child confirmed that there were the same amount of cubes in both rows. Following this,one of the rows was elongated and the child was asked: **Now do you think that there is one row where there are more or do you think that the two rows have the same amount of cubes?**

Of course, the elongation task proved to be much easier than the Piagetian test. All the 13 children who perceived the invariance of quantity with respect to dispersion, contraction, displacement, and rotation, also perceived its invariance with respect to elongation. Of the five children in transition, only one succeeded on the elongation task; two of the five children who did not perceive the invariance with respect to dispersion and contraction did succeed. Thus in total, 13 subjects were successful on this task. On this basis, this invariance is barely more accessible than that due to dispersion.

The Piagetian test was more difficult since the mere visual perception of the two rows of different lengths creates a cognitive conflict, the child often believing that the elongated row must now contain more elements. Ten of our 24 subjects (42%) were judged to conserve quantity. Eight of these belonged to our 13 students who had been successful on every prior invariance task except the ones on visual perception of the units. But surprisingly one student came from the transitional group and one from the group that did not perceive the invariance of quantity with respect to configuration. Another important difference was to be found between the two age groups: only 3 out of 8 children (27%) in the younger age group conserved quantity while 7 out of 13 in the older age group (54%) did so too.

By way of conclusion

These research results bring out the fact that the notion of discrete quantity exists in the child's mind independently of numeration. In fact, the data shows that the kindergartners' conception of discrete quantity is quite extensive and that they have non-numerical procedures to deal with many of the related problems. Of course, the results also show that the abstraction of quantity, that is, the perception of its invariance with respect to surface transformations, is an ongoing process. The various tasks we have designed provide sufficient information to establish a hierarchy among these different invariances. The following hierarch is based on the success rate of the above tasks:

Invariance of quantity with respect to	N	Percentage
rotation (of plate with cubes)	19	79%
displacement within same space	17	71%
elongation of a row	16	67%
random contraction of a set	16	67%
random dispersion of a set	15	63%
comparison of elongated row (Piaget test)	10	42%
the visual perception of objects (in the bag)	3	13%
the visual perception of objects (hiding cardboard)	1	4%

The information communicated in this paper deals with the notion of discrete quantity, a pre-numerical concept which is the foundation of cardinality. An equally important pre-numerical concept is that of the **rank of an element** in an ordered set. That particular aspect is presented in a companion paper, **The kinfergartners' understanding of the notion of rank**, by N. Herscovics and J.C.Bergeron.

References

Bergeron,A., Herscovics,N., Bergeron,J.C.,(1986), Counting tasks involving some hidden elements, in **Proceedings of the Eighth Annual Meeting of PME-NA**, the North American Chapter of the International Group for the Psychology of Mathematics Education, Lappan,G. & Even,R. (eds.), East Lansing, Michigan: Michigan State University, 21-27

Freudenthal, H., (1983), **Didactical phenomenology of mathematical structures**, Dordrecht: Reidel Publishing

Piaget,J., & Szeminska, A., (1941/1967), **La Genèse du nombre chez l'Enfant**, 4^{e} éd., Neuchatel, Suisse: Delachaux et Niestlé

Steffe,L.P., von Glasersfeld,E., Richards,J., Cobb, P., (1983), **Children's Counting types**, New York: Praeger Publishers

Vergnaud, G., (1979), The acquisition of arithmetical concepts, in **Educational Studies in Mathematics**, 10, 263-274

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A REVIEW OF RESEARCH ON VISUALISATION IN MATHEMATICS EDUCATION

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ABSTRACT

The aspect of visualisation in Mathematics education has not attracted much research attention in the recent past. Nevertheless it is felt by many Mathematics educators to be important in the education process. There have been some significant studies which do have interesting implications for both research and practice, and this review surveys the current situation. The review is in three sections, the visualisations themselves, the process of visualisation, and teaching in relation to visualisation.

1. INTRODUCTION

This review builds on and extends from earlier reviews written either by the author or by others (Bishop, 1980; Bishop, 1983; Bishop, 1986; Clements, 1982; Presmeg, 1986b; Mitchelmore, 1976) but will be restricted to the notion of 'visualisation'. This construct interacts in the research literature with the ideas of imagery, spatial ability, and intuition, but it is certainly not the case that visualisation has been felt to be a significant research area in mathematics education in the recent past. Whilst searching the literature in preparation for this review, it was surprising to discover that in the J.R.M.E. listing of 223 research articles in 1985 only 8 were remotely connected with the topic, that in the same listing for 1986 only 7 out of the 236 articles were related and at PME XI no papers were specifically focussed on visualisation in mathematics education.

2. THE 'OBJECTS' OF VISUALISATION

Mathematics is a subject which is concerned with objectivising and representing abstractions from reality, and many of those representations appear to be visual i.e. they have their roots in visually-sensed experiences. These visualisations may be relatively primitive, i.e. imagining a particular door handle being rotated, or they may already be relatively abstract, i.e. an imagined right angled triangle inscribed in a circle. They are clearly of significance in mathematical activity, as witnessed by writers such as Hadamard (1945) and they have been of interest to some researchers for many years.

The first point of interest is that visualisations (as I shall term these phenomena) are a very individual matter. There is a wide range of visual imagery used by individuals even when restricted to mathematical activity. Presmeg (1986b) lists five different kinds of visual imagery which she identified in her students:

1. Concrete, pictorial imagery (pictures-in-the-mind);
2. Pattern imagery (pure relationships depicted in a visual-spatial scheme);
3. Memory images of formulae;
4. Kinaesthetic imagery (involving muscular activity e.g. fingers 'walking');
5. Dynamic (moving) imagery.

Moreover the students did not stay with only one of those types, but used different ones in different situations.

The range of visualisations generated by individuals is therefore an important factor to keep in mind. The quality of the visualisations generated also appears to vary in a marked way, with 'vividness of imagery' being a favoured construct (see, for example, Richardson, 1977 and Sheehan, 1966). Presmeg (1986a) for example, found that 'vividness of images' did help her students, particularly in memory situations: "Memory images of formulae, and pattern images, are two types of imagery which provide a quick means of recall of abstract general principles and procedures, the former in a concrete image which encapsulates a procedure, the latter in a more schematic image which stressed regularities" (p.301).

Other qualities of visualisations often appear in a negative frame, relating more to the obstacles which they can create. For example, Hoz (1981) refers to what he calls "geometrical rigidity" caused by a child being unable to 'see' a diagram in a different way. Related to this is the case where the orientation of the shape is tied too firmly with the shape itself, and for some children it really is a challenge to draw an isosceles triangle which is also right-angled. Fischer's (1978) research suggests that the preference for 'upright' figures is very deep-seated, and appears not to be affected by particular kinds of instruction.

Other familiar examples of problems caused by the rigidity and symbolisation of visualisations are illustrated in the research of Hart (1981), Kent (1978) and Kerslake (1979).

Presmeg (1986b) summarised these kinds of difficulties experienced by the 'visualisers' in her study as follows:

1. the one-case concreteness of an image or diagram may tie thought to irrelevant details, or may even introduce false data,
2. an image of a standard figure may induce inflexible thinking which prevents the recognition of a concept in a non-standard diagram,

3. an uncontrollable image may persist, thereby preventing the opening up of more fruitful avenues of thought. (This difficulty is particularly acute if the image is vivid.)
4. especially if it is vague, imagery which is not coupled with rigorous analytical thought processes may be unhelpful.

However she also comments on the power of the generalised graphic schemes recognised by Krutetskii (1976) and the pattern imagery illustrated by the work of de Groot (Harris, 1980). She shows in her transcripts the positive value felt by pupils of having certain kinds of visualisations available.

We have clearly moved on in our knowledge from merely believing that *all* visualisations play a useful role in mathematical activity, to understanding something of their features which contribute significantly to that role. It seems therefore that more attention needs to be paid in research to the particular qualities of visualisations in order to understand more about which visualisations are more helpful than others in a given mathematical situation.

3. THE VISUALISATION PROCESS

The visualisations to which we have been referring don't just happen by accident. The process of visualisation in mathematics is recognised as a complex one but one which is important to try to understand. Once again the focus has been on the individual nature of this proces,and recently there has developed a strong research interest in learners who seem to excel in it. The so-called 'visualisers'- those problem-solvers who prefer to use, and use well, visual processing - are now a well-studied group. Krutetski (1976), Moses (1977), Presmeg (1986b), Lean and Clements (1981) and Suwarsono (1982) are just some of the researchers who have focussed on this area, and have helped to move the field away from the rather sterile factor-analytic and psychometric studies of spatial ability which characterised much of the earlier research. It is perhaps just worth noting, in passing, that it was the study of Guay et al. (1978) which finally convinced many people that the psychometric approach was inappropriate for studying the visualisation *process*.

But what then can we learn about the process of visualisation?

> Presmeg (1986b) states "A visual method of solution is one which involves visual imagery, with or without a diagram, as an essential part of the method of solution, even if reasoning or algebraic methods are also employed".
>
> Moses' (1977) 'degree of visuality' score was based on "the number of visual solution processes (e.g. pictures,

> graphs, lists, tables) present in the written solutions".
>
> Krutetskii's (1976) 'geometric type', "felt a need to interpret visually an expression of an abstract mathematical relationship and demonstrate great ingenuity in this regard".
>
> Suwarsono's (1982) visuality score was high "if the correct answer was obtained and reasoning was based on a diagram (drawn by the pupil) or on some ikonic visual image (constructed by the pupil)".

At its simplest then, the visualisation process appears to involve the learner constructing some kind of visualisation and using it appropriately. Let us analyse this further, though. In the problem- solving situation, 'appropriately' clearly means 'to help obtain a solution'. It is surely helpful however to have a broader notion than just 'problem-solving' because obviously the visualisation process needs some sort of trigger or stimulus, so different tasks will stimulate different images. For example, a task such as "Find as many figural representations of 2x3=6 as you can", is likely to evoke a very different response from a problem like "What face of the dice is on top if the 2 is facing you, the 3 is on the right and the 6 is at the bottom". Not any visual image will do for this purpose, so once again it needs to be an 'appropriate kind of visualisation' - we saw in the previous section some of the negative effects of particular visualisations.

One implication of this analysis is clearly that if we want to understand more about the visualisation process, we need to study it in a variety of task and stimulus contexts, and to move away from just 'problem-solving'. One hint from Presmeg's (1986a) study is "Apparently when a topic is first taught, a visual presentation often aids visualisers' understanding, but practice of the procedure or formula may lead to habituation when an image is no longer necessary. In other words, facility led visualisers away from visual methods". This aspect is also referred to by Lean and Clements (1981) in relation to the Moses study (1977) in which most of the problems used were too difficult for almost all the students (p.272). Clearly ease or difficulty of the task is one feature of the stimulus context. It would therefore be interesting to discover just how stable across tasks and other contexts the construct of 'visualiser' is. The assumption of 'once a visualiser always a visualiser' is clearly being challenged. What is important for research however is not how valid or reliable is the label, but what do we learn from such children about the visualisation process. That is what attention now needs to be firmly directed towards.

4. VISUALISATION IN EDUCATION SITUATIONS

This the third aspect is intended to focus attention away from the predominantly individual considerations of the first two sections of this review. Some studies fall into the category of 'training' and Lean (1981) has summarised these (reported in Bishop, 1983). He concluded: "The evidence...indicates that these various skills (involved in interpreting figural information) are trainable given the appropriate experiences". He did not however find any evidence showing the success of training in visual processing. Indeed this is not surprising in view of the highly individual nature of visualisation which has been found. What, one might ask, can 'training' mean in this context?

Of more interest are studies like Mitchelmore (1980 and 1984), Marriott (1978) and Bishop (1973) which deal in different ways with aspects of the material environment which interact with visualisation. For example, Mitchelmore (1984) interpreted his findings of relatively weak spatial and visualising skills amongst his learners in Jamaica in this way: "Many homes in Jamaica lack special play equipment for children. They have fewer toys...The effect of such a home environment is dramatized by the exceptions that come to light...such as the grade 4 son of a mechanic with a workshop at his house and the grade 6 boy who often helped his mason father; both boys did outstanding work on symmetry in rural classes consisting mostly of farmers' children" (p.139).

From studies like these there is some evidence that a learning environment in which structured and manipulative materials predominate can help to encourage the creation of visualisations and thus the visualisation process itself. This kind of research is being up-dated by current studies of the influence of computer environments (see for example, Noss, 1987 and Hoyles, 1987).

Mention of the teacher there reminds us that there is a strong role to be played by the social, as well as by the material, environment. For example in all the studies reported so far in this section, the role of the teacher (or in one case the two parents) has been assumed as benign. Certainly the teaching shown in Kent and Hedger's (1980) study appeared to be very helpful, from the perspective of visualisation.

However, Presmeg's (1986a) study was much more informative because it focussed as much on the teachers as on the pupils. By using the same tasks to assess the visuality of the teachers as she did for that of the pupils, she grouped the teachers into three types, and analysed their teaching styles.

Of particular interest was how these different teachers interacted with the 'visual' pupils. She says this: "In the classes of teachers in the non-visual group, it was found that non-visual teaching had the effect of leading visualisers to believe that success in mathematics depended on rote memorisation of rules and formulae". "Visualisers in classes of teachers in the middle group appeared to benefit from a teaching stress on abstraction and generalisation which was associated with this group of teachers. Pattern imagery and rapid use of curtailed methods were encouraged in the thinking of visualisers with these teachers." "Teachers in the visual group were unanimously positive in their attitudes towards visual methods, but they were not always able to lead visualisers to overcome the difficulties, and to make optimal use of the strengths, of visual processing" (pp.308-9).

It is clear that because visualisation is such a personal and individual matter, the teacher's role is a subtle one. Certainly there need to be many more studies which take the teacher's visuality into account, which look in detail at the kinds of teaching which the teacher's own visual processing develops, and which examine the effects of these on the individual pupil's processes.

BIBLIOGRAPHY

Bishop, A.J.: 1973, 'Use of structural apparatus and spatial ability: a possible relationship' Research in Education, 9,43-49.

Bishop, A.J.: 1980, 'Spatial abilities and mathematics education - a review' Educational Studies in Mathematics, 11,257-269.

Bishop, A.J.: 1983, 'Space and Geometry'. In R.Lesh and M.Landau (eds) Acquisition of Mathematics Concepts and Processes, Academic Press, New York.

Bishop, A.J.: 1986, 'What are some obstacles to learning geometry?' Studies in Mathematics Education, UNESCO, Vol.5, 141-159.

Clements, K.: 1982, 'Visual imagery and school mathematics' For the Learning of Mathematics, 2,2 and 3.

Fisher, N.: 1978, 'Visual influences of figure orientation on concept formation in Geometry'. In R.Lesh (ed.) Recent Research Concerning the Development of Spatial and Geometrical Concepts, pp.307-321, ERIC/SMEAC, Columbus, Ohio.

Galton, F.: 1883, Inquiries into the Human Faculty and its Development, Macmillan, London.

Guay, R.B., McDaniel, E.D. and Angelo, S.: 1978, 'Analytic factors confounding spatial ability measurement'. In R.B.Guay and E.D.McDaniel (eds) Correlates of performance on spatial aptitude tests. US Army Research Institute for the Behavioural and Social Sciences, Purdue University.

Hadamard, J.: 1945, The Psychology of Invention in the Mathematical Field, Dover, London.

Harris, L.J.: 1980, 'Sex differences in spatial ability: possible environmental, genetic and neurological factors'. In M.Kinsbourne (ed) Hemispheric Asymmetries of the Brain, Cambridge University Press, Cambridge.

Hart, K.M.: 1981, Children's Understanding of Mathematics: 11-16, John Murray, London.

Hoyles, C.: 1987, 'Geometry and the computer environment'. In J.C.Bergeron, N.Herscovics and C.Kieren (eds) Proceedings of the Eleventh International Conference of P.M.E., Montreal,

BEST COPY AVAILABLE

Canada.

Hoz, R.: 1981, 'The effects of rigidity on school geometry learning' *Educational Studies in Mathematics*, 12, 171-190.

Kent, D.: 1978, 'The dynamic of put' *Mathematics Teaching*, 82,32-36.

Kent, D. and Hedger, K.: 1980, 'Growing Tall', *Educational Studies in Mathematics*, 11,137-179.

Kerslake, D.: 1979, 'Visual mathematics' *Mathematics in School*, 8,34-35.

Krutetskii, V.A.: 1976, *The Psychology of Mathematical Abilities in Schoolchildren*, University of Chicago Press, Chicago.

Lean, G.A.: 1981, Spatial training studies and mathematics education. Unpublished paper, University of Cambridge, Department of Education.

Lean, G.A.: and Clements, M.A.: 1981, 'Spatial ability, visual imagery, and mathematical performance' *Educational Studies in Mathematics*, 12,267-299.

Marriott, P.: 1978, 'Fractions, now you see them, now you don't'. In D.Williams (ed.) *Learning and Applying Mathematics*, Australian Association of Mathematics Teachers, Melbourne.

Mitchelmore, M.C.: 1976, 'Cross-cultural research on concepts of space and geometry'. In J.L.Martin (ed.) *Space and Geometry*, ERIC/SMEAC, Columbus, Ohio.

Mitchelmore, M.C.: 1980, 'Three dimensional geometrical drawing in three cultures' *Educational Studies in Mathematics*, 11,205-216.

Mitchelmore, M.C.: 1984, 'Spatial ability and geometry teaching in Jamaica' *Studies in Mathematics Education*, UNESCO, 3,135-143.

Moses, B.E.: 1977, The Nature of Spatial Ability and its Relationship to Mathematical Problem Solving. Unpublished Ph.D. dissertation, Indiana University, Bloomington.

Noss, R.: 1987, 'Children's learning of geometrical concepts through LOGO' *Journal for Research in Mathematics Education*, 18,343-362.

Presmeg, N.C.: 1986a, 'Visualisation and mathematical giftedness' *Educational Studies in Mathematics*, 17, 297-311.

Presmeg, N.C.: 1986b, 'Visualisation in high-school mathematics' *For the Learning of Mathematics*, 6,3,42-46.

Richardson, A.: 1969, *Mental Imagery*, Routledge and Kegan Paul, London.

Sheehan, P.W.: 1966, 'Functional similarity of imaging to perceiving: Individual differences in vividness of imagery' *Perceptual and Motor Skills*, 23,1011-1033.

Suwarsono, S.: 1982, Visual Imagery in the Mathematical Thinking of Seventh-Grade Students. Unpublished Ph.D. dissertation, Monash University, Melbourne.

ACQUISITION OF MEANINGS AND EVOLUTION OF STRATEGIES IN PROBLEM SOLVING FROM THE AGE OF 7 TO THE AGE OF 11 IN A CURRICULAR ENVIRONMENT

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This report deals with the aim of observing the development of the ability to solve arithmetical (word) problems from the age of 7 to the age of 11 in a "curricular environment"(a complete teaching project concerning all subjects taught in the Italian primary school).Regarding the solving of arithmetical problems in particular,the project focusses on the development and use of verbal language,and the development of problem-solving strategies by means of the resolution of arithmetical problems without numerical data in various fields of experience and through the gradual progression from spontaneous calculation strategies to standardised calculation procedures.One reason why this report may be of interest is that concerning certain crucial questions related to problem solving,it allows comparisons to be made with research carried with different methodologies in different experimental situations.

1.INTRODUCTION

This report deals with certain aspects of the development of the ability to solve arithmetical (word) problems in children aged 7 to 11.It studies in particular the relationship between the nature of the problems proposed (physical variables,...) and the types of reasoning which children employ in order to solve them,the relationship between the planning of the sequence of operations to be carried out and the calculation strategies conducted on numerical data,and the relationship between the development of verbal abilities and the working out of strategies.

In what will of necessity be a somewhat schematic form,this report proposes to point out certain phenomena which were noticed in a "curricular environment":we have followed children as they progressed from the age of 7 to the age of 11,in the classroom,while experimenting a project which dealt with all subjects taught in the Italian primary school(particular attention being paid to linguistic education).In this sense,the report provides elements for comparison with many other articles on problem solving. Most of these articles are based on systematic,very accurate,but short-term observations,which often are carried out in the context of a curriculum on

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which the researcher has no influence and of which he does not even know all the details (this applies particularly to subjects other than mathematics).

Paragraph 2 will outline the situations in which the children were observed;it also explains the methods of observation used,and the tools employed to make a comparison with a reality much vaste than the one we analysed,and to which this report refers.

Paragraph 3 presents the questions with which the classroom observations were concerned,the analysis of the informations gathered and some conclusions which emerged from this work.

Paragraph 4 contains a description of some particular examples of problems,which I regard as typifying the conclusions presented in paragraph 3.

2.THE EXPERIMENTAL SITUATION

The observations which form the basis of the present report were conducted within the framework of a complete teaching project related to all subjects in the Italian primary school (6 to 11 years).The planning of this project began in 1977,and the project itself was put into effect in 1980/81.This project involves now 120 classes.The teacher's work is guided by detailed outlines of the content of the work to be done in the classroom,and how to handle it.Starting with the second class (7 years),the children are given guided worksheets on the various teaching units and the various subjects which the project is concerned with.The extent of the worksheets is increased gradually.For every class there is a standardised test(based on "open" questions) half-way through the year and at the end of the year.

This report refers particularly to four classes under observation:two of which were followed between the ages of 6 and 11,and two between the ages of 6 and 10.As far as problem solving is concerned,the methods of observation and analysis of the children's performance employed were the following:

- a systematic gathering of the original texts written by the children on all the problems that they worked on individually (about 20 each year in each class)
- periods of observation (from 5 to 10 in the course of each year) by a researcher present in the classroom during the solving of particular problems.
- taped recordings(carried out by the teacher or by the researcher during his periods of observation) of interaction between the children or of in-

teraction with the teacher.

- sporadic observations and reports by the teachers.

As for comparison with the other classes involved in the project,this was achieved by means of the standardised tests conducted half-way through and at the end of the year,and by collecting the written texts of individual children on particularly interesting problems,which had been pointed out to the teachers in advance.

With regards to the matters discussed in paragraph 3,the following didactic choices made for the classes under observation seem to be relevant:

- the majority of the problems presented to the children in teaching units between the ages of 6-7 to 11 were related to economic matters,natural phenomena (like shadows),or the depiction on paper of real spacial situations (topographic maps,etc).
- some of the problems presented to the children did not contain numerical data:the children were asked to plan a strategy to solve the problem
- in the majority of cases the problems are handled in the classroom in the following way:first the children work individually,then they compare the strategies they adopted to solve the problem
- before the children are taught written techniques of arithmetical calculation the teacher points out some general procedures which are based on the calcultation strategies employed spontaneosly by the children
- from the age of 6-7 years,particular emphasis is placed on the ability to report in writing about the processes and experiments conducted by the class,about the solving of mathematical problems,and about the procedure for handling geometric and graphic work.

3.THE QUESTIONS EXAMINED

Further on,when I speak of "solving strategies" ,I will be referring to the sequence of operations chosen by the pupil in order to solve an arithmetical problem (including possible practical techniques involving coins or strips of paper or the making of graphs,etc.).When I speak of "calculation strategies",I will be referring to activities involving numbers,which produce intermediate numerical results or a final solution of a problem. When I speak of "calculation procedures" I will mean the making of calculations following routine procedures which have already been explained to the pupils,and which they recognise.

It is well-known that in many cases it is difficult to distinguish bet-

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ween these elements involved in the solving of arithmetical problems.However,there are arithmetical problems,which are widely employed in our classes,in which it is possible to induce the children to concentrate their attention on solving strategies or on calcultation strategies (see par.4)

3.1.Experience fields and strategies employed:regarding the solving strategies and the calculation strategies employed,what is the role of the "experience field" proposed by the teacher (economic problems,problems of time span,etc) for the development of the skills involved in the solving of arithmetical problems ?As the examples in paragraph 4 will show in more detail, regarding the relevance of the "experience field"in helping the child to understand the meaning of certain arithmetical operations and solving strategies,our observations tally with the results of other research that has been carried out. The contribution that certain "experience fields" can make to the development of calculation strategies also seem to be relevant. This applies both to the properties of the operations implicitely employed by the child,and to the identification of general calculation procedures. One interesting result of our observations is that when it comes to making implicit use of the properties of the operations ,the child immediately seems to be able to perform the jump from problems solved in certain "experience fields" to mental calculations with "pure" numbers.On the other hand, the child does not seem to be able to make the jump spontaneously from one "experience field" to another (this confirms the importance of the "experience field" in stimulating certain types of behaviour).Finally,we observed that within a single"experience field",with the same physical variables and analogous numerical values,the nature of the problem proposed can lend to various strategies depending on the way the child perceives the problem.

3.2.Solving strategies and calculation strategies:is there any consistency between the first and the second ? In particular,are the meanings of the operations involved in the choice of operations,consistent or not with the meanings of the operations involved in the calculation strategies used in the operations chosen(before the children have acquired the techniques of written calculation) ? Our observations would seem to indicte that such consistency is not always to be found,but that in calculation strategies, alongside the meanings of physical variables expressed by numerical data,an important role is played by the particular numerical values assigned and by the experience in the use of calculation strategies acquired in other circumstances.In particular,we observed inconsistencies in connection with problems of subtraction(which,at a numerical level,were often solved by means of a "completion",irrespective of the fact that the problem calls for "taking away"),and above all with problems o division (in many cases,in situation calling for a "subdivision into equal parts",we observed calculation strategies based on comparisons and content-container relationships).It is possible that these inconsistencies derive from the fact that in our project problems of"completion" and problems of division of homogeneous quantities are dealt with before the other problems of subtraction and division respectively.The inconsistencies pointed out do not have any harmful effect as far as the ability to solve problems is concerned,but they make difficult to provide the child with a formal description of the way he reasons when he tries to solve a problem:" 4500-1850 = ..."is a good formula to describe the choice of operation required to solve a "remainder" problem,but a good description of the calculation strategy adopted by many children would be : " 1850 + ... = 4500 " .

3.3.Calculation strategies and calculation procedures: what degree of autonomy and awareness it is possible for children aged 7 to 11 to reach regarding the shift from spontaneous calculation strategies designed to solve particular problems,to general calculation procedures ?We carried out observations and experiments concerning the four arithmetical operations,paying particular attention to the working out of a written calculation technique for division.We think we are justified in concluding that in the classroom it is not difficult to elicit (by proposing suitable problems with suitable numerical data) spontaneous calculation strategies which are convenient for application in universal calculation procedures;however (at least until the age of 9-10) the pupils do not seem capable of evaluating by themselves which of the resulting strategies is the most suited to being transformed into a general,efficient calculation procedure.Moreover,it sometimes happens that the procedures developed in this way in the classroom are not in accordance with traditional standard procedures (see 4.3).

3.4.Verbal language and problem solving: what is the role of verbal language in the solving of arithmetical problems ? In our classes we have the possibility to influence the development of linguistic skills,since,from the age of 6 to 11,the teacher is the same.We are thus able to observe how,as far as problem solving is concerned,such an influence results in different behaviour and performance compared to classes which follow a traditional curriculum of linguistic education.In all our classes each child is normally

required to illustrate the solution he has worked out.The children are also advised to write down what they think while they are trying to solve a problem.In some classes of children aged 6 to 11,particular attention is paid to the activity of verbalising work produced in class.This may involve discussions of how pocket calculators work,or an explanation of automatic processes,etc.These verbalisation activities share the characteristic of requiring the child to develop an expository language whose logical organisation is linked to that of external object to which the subject refers.A comparison between classes which have undertaken these activities extensively and classes which have not,reveals a significant difference in the ability to produce a written solution to the problem.This applies above all to the more complex problems,or to arithmetical problems without numerical data.Such differences were observed not only in the quality of the expository text produced,but also in the ability to produce such a text,and thus in the ability to solve a problem.However,regarding the use of verbal skills in the working out of calculation strategies,the differences between the two groups of classes were much less striking.In classes which engaged thoroughly in the non-mathematical verbalisation activities described above,we noticed that some children use verbal language extensively in the working out of calculation strategies too,since they prefer this to the formalism of algebra or the language of graphs.However,many children in these and in the other classes seem reluctant to verbalise the working out of calculation strategies,as if in their mind the rythms of reasoning about numbers were "out of phase" with the requirements of verbal text production.
Experience of verbalisation in a non-mathematical context in situations of "bound logic" creates a great difference in the ability to identify and declare which characteristics of a given calculation strategy are suitable for general adoption.

4.SOME EXAMPLES OF PROBLEMS ANALYSED

Given the limited space at my disposal,it is not possible to give a detailed illustration of one or more of the problems which were used as the basis of our classroom observations.Thus I will confine myself to describing, in a very concise way,some problems related to the points considered above. I must emphasise however,that each of them forms part of a work programme concerned with the same "experience field" and that such a programme usually contains many problems which are presented over a number of years.

4.1.Subdivision of lengths: the following problems are usually proposed to nine-year-old children:

a) the children are required to plan the depiction of a period of twenty centuries along one of the classroom walls,in such a way that for each century there is available a "space" equal in length to that which is reserved for the other centuries.

b) the children in the class (usually,between 18 and 25) are required to spread out at an equal distance from each other around a circle drawn on the floor of the gymnasium,or along a wall in the gymnasium.

For a) there are two types of solving strategy : in the first one,the children plan to measure the lenght of the wall,then to divide this lenght into 20 equal parts,etc. In the second type of strategy,the children plan to place a strip of paper or string on the wall and then to lenghten it or shorten it "until it fits exactly 20 times".

For b) (whether one is dealing with a circle drawn on the floor,or with a wall in the gymnasium;and I must point out that the children have never measured in the classroom the circumference of a circle before) the vaste majority of the strategies adopted involves dividing the length of the wall or circumference of the circle into as many equal parts as there are children in the class.Even the children who have started with the idea of the "equal space between one child and another",spontaneously after ask themselves "how much space is there for each child".

These two problems are presented ,at the same time,to classes which have previously carried out very similar activities:the difference in strategies adopted is very striking .As we saw in 3.1.,the nature of the problem and the way the child perceives it seem to be important factors in distinguishing between strategies.As was made clear in 3.4.,as far as both a) and b) are concerned,striking differences are to be found between the success rates of classes which are accustomed to extensive "bound logic" verbalisation activities and classes which are not.

4.2.Distributive property in multiplicative problems :let us consider the following two problems which were presented to children aged 7-8 before they had learnt multiplication (written) technique:

c) the purchase of five objects which cost 310 liras each

d) the calculation of the total length of a strip of paper consisting of five pieces,each 310 centimetres long.

These problems are also presented at the same time to classes which have for the most part followed the same teaching programme on word problems.

In the first case,most of children decide to group together the hundred lira coins(" which makes 1500 liras") and add the five 10 lira coins.Whereas in the second case,most of the calculation strategies consist of making addition in columns (or in lines): "310 and 310 makes 620,add 310,makes 930...". As we saw in 3.1., the calculation strategy seems to be heavily influenced by the nature of the variables involved ,and by previous experience(practical and numerical) of problems assigned in the same "experience field". The jump from the field of operations with numbers involved in problem a) to the field of "pure" numbers (mental calculation) is made spontaneously by many children.

4.3.From problems of division to the technique for the written calculation of divisions:when the children in our classes are 8-9 years old,we begin work which in 3-4 months will lead to a technique for the written calculation of divisions,consciously worked out by the children themselves on the bases of spontaneous calculation strategies used in problems of division with suitable numerical data.Examples of strategies worked out :

e) with sums of money such as 12000 liras to be subdivided among,say,four children,many children spontaneously reason in this way: "one thousand for the first,one thousand for the second...."(notice that this strategy is never suggested by our teachers;however it is present in the children's activities outside the school environment)

f1) successive approximations:being required to divide 37500 liras among 16 children,many children "try for":1000 liras each...2000 liras each... 3000 liras each...(too much!),alright,I'll try 2100 each...2200 each..."

f2) with the same sum of money and the same number of children,other children (a minority) "try for": "1000 liras each...2000 liras each...3000 liras each...(too much!) ",so they subtract 32000 liras from 37500 liras and carry on,dividing the remaining 5500 liras among the 16 children,etc.

It is not easy for the children to realise by themselves the superiority of the calculation strategy f2) with a view to a universal procedure for the calculation of divisions.Moreover,it must be pointed out that such a universal procedure is not the standard one taught in the majority of countries.It should be noted also how the children handle a typical problem of "sharing out " by reasoning about "content-container relationship" in the moment when they work out the calculation strategy.

REFERENCES : see : LESH,R. - "Conceptual Analysis of Mathematical Ideas and Problem Solving", PROCEEDINGS P.M.E. 1985.pp.73-96

THE RELATIONSHIP BETWEEN CAPACITY TO PROCESS INFORMATION AND LEVELS OF MATHEMATICAL LEARNING

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This paper is a description of studies between 1985 and 1987, of the relationship of levels of capacity to process information to mathematical knowledge in rural-urban and urban Australian Aboriginal and non-Aboriginal children, aged 3-8 years. The 1985/86 results for rural-urban Aboriginal children are discussed in this paper. The 1987 data will be presented at the conference. Capacity to process information was measured by four tasks; two that allowed comparison of results with large studies and other populations and two that assessed the same levels of capacity but utilized knowledge systems familiar to Aboriginal children. Tests were also designed to measure basic length and number tasks at increasing levels of demand on processing capacity. In 1987 formal school mathematics learning was also measured. Measures of capacity were significant predictors of performance on length and number concepts. The performance with age of Aboriginal children on capacity and basic length and number tasks was comparable with non-Aboriginal children. The results indicate that lower school achievement by these children cannot be attributed to lesser capacity to process information or to inability to cognize basic mathematical concepts. Other factors that probably affect school achievement are discussed briefly.

This paper is concerned with one important factor in learning, that is increase in capacity to process information, and the extent to which it determines the complexity of mathematical concepts that young Aboriginal and non-Aboriginal children should be able to cognize.

Cognitive theorists such as Case (1985), Fischer (1980) and Halford (1982) have argued, from different theoretical perspectives, that there is an upper limit to children's capacity to process information which increases with maturation and learning. Case (1985) described measures and norms for short term storage space (STSS) and demonstrated that STSS increased with age. He postulated that increase in STSS would determine the complexity of concepts that children could learn. Fischer (1980) described the construction and control of hierarchies of skills that he maintained are domain specific and limited by increasing capacity. Halford (1982) proposed three levels of thinking that depend on children's increasing capacity to match systems of symbols to elements in the environment. At level 1 children should be able to cognize binary relations, at level 2 binary operations and integrations of relations such as those required in transitive reasoning, and at level 3 compositions of binary operations. Halford has identified examples of thinking at each of these levels, on the basis of empirical evidence and analysis of demand. Most of the tasks in this research have been designed or analysed to measure the first two levels of thinking postulated by Halford.

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There have been recurring debates in the literature of culture and cognition (cf. Laboratory of Comparative Human Cognition, 1983) as to whether there are cultural differences in cognition as such or whether culture and cognition form an interacting system which produces context specific differences in performance. The hypothesis in this research is that the latter is the case. It is assumed (cf. Fischer, 1980) that capacity to process information is applied selectively to specific concepts within a domain depending on motivation, experience and knowledge. This should mean that a child in a particular cultural setting will only perform to capacity on some of a possible set of concepts that make the same cognitive demand. It should be possible to predict however that if a child succeeds on one task at a particular level of demand then s/he has the capacity to cognize an isomorphic task, given sufficient motivation and experience. Boulton-Lewis (1983, 1987a) for example found with a sample of non-Aboriginal Australian children, aged 3 to 7 years that as capacity to process information increased so generally did knowledge of components of length measuring.

Australian Aboriginal children, including those living in urban environments, are usually not as successful with formal school learning as non-Aboriginal children (Bourke and Parkin, 1977; Seagrim and Lendon, 1980; Watts, 1976). It was known that the children tested in 1985 and 1986 (Boulton-Lewis et al. 1986, 1987b, paper submitted) did not perform as well as non-Aboriginal children on mathematical and other tests in the final year of primary school. Differences in cognition and school learning in Aboriginal children have been variously attributed to differences in cognitive style (e.g. Watts, 1976), processing strategies (Kearins, 1976; Klich and Davidson, 1984) or environmental factors rather than intellectual capacity. This research addressed the hypothesis that Aboriginal children have the same capacity to process information as any other child but for social or cultural reasons perhaps do not learn to apply that capacity to school mathematics.

THE RESEARCH PROGRAM

Samples

A preliminary study using four capacity measures was carried out in 1985 with a sample of 20 children of mean age 6 years (Boulton-Lewis et al. 1986). In 1986 a sample of 75 children from Cherbourg, aged 4-8 years, was tested for processing capacity and basic mathematical concepts (Boulton-Lewis et al. 1987b; paper submitted). Children in both samples attended school at Cherbourg.

Cherbourg is an Aboriginal community just out of Murgon, which is a large rural town about 290 km north of Brisbane. The community was established in 1905 (Koepping, 1977). The population has been described as a high contact Aboriginal group (McElwain and Kearney, 1973). Most of the children in the samples were 3rd or 4th generation residents. The people of the community originally came from different language groups but now most speak standard English with some local dialectical variations.

In 1987, 60 urban Aboriginal children aged 4-8 years and a matched sample of 60 non-Aboriginal children, all attending Brisbane schools, were tested with the same measures of capacity and basic mathematical concepts and with additional tests of school mathematics learning.

Tests

1. Capacity Measures

   Because this research was conducted in a cross-cultural situation two of the capacity tests were chosen to allow comparison of results with those obtained with large samples of non-Aboriginal children elsewhere (viz. Cucui, cf. Case, 1985, and the Matrix Task, Halford, 1980). Two other capacity measures were designed to measure levels of capacity process aspects of familiar knowledge systems. The Playing Card Relations test measured levels of processing of knowledge of playing cards. The Family Structure test relied on children's knowledge of family relationships (usually a matter of importance to Aboriginal people). All of these tests are described in detail elsewhere (Boulton-Lewis et al. 1986, 1987b, paper submitted). The Family Structure test is described briefly below as an example.

The critical feature of transitive reasoning is the ability to integrate relations. The Family Structure test was designed to measure capacity to cognise binary relations and then to integrate these. For example at level 2 of this test the ability to explain that, if John is the son of Mary, and Mary is the sister of Jane, then John is the nephew of Jane, entails integrating relations and measures the same essential cognitive process as reasoning that if a>b, and c<b, then a>c. Thus the cognitive process of transitive reasoning was measured using a familiar knowledge system. At level 1A of the same test, after discussion of pictures of a family, children were merely required to recall names and positions of family members e.g. "That's Mary". "She's the Mother". At level 1R children were required to describe family relationships, in response to questioning, in terms of a single binary relation, e.g. "This is John. He calls Hope his sister because they have the same mother and father/family."

2. Length and number tasks.
Tasks were devised to test knowledge of length and number which required responses demanding reasoning at level 1 (nominal knowledge), level 1R (binary relations) and level 2 (integration or composition of relations). Children were asked to name, compare, order and seriate lengths. Finally they were asked to determine comparative lengths of configurations on the basis of the size and number of component units. Similarly children were tested for ability to subitize sets with from 2-4 members (level 1), order pictorial representations of sets (e.g. 3,4,5,6 objects) by pair by pair comparisons (level 1R) and to perform the operations of addition, subtraction and reason transitively (level 2). These tests are all described in detail in Boulton-Lewis, et al. 1986, 1987b, paper submitted. Finally in 1987 children were tested for written symbolic (generally school learned)

knowledge of the concept tests described above.

Testing Procedure

The testing method was a kind of clinical interview. Children were tested one by one, mostly by trained Aboriginal people. They were asked to respond to materials and pictorial representations. Practice was included prior to all the tests. Some of this was quite extensive as in the Matrix Completion task (Halford, 1980). In that test children were given up to 16 opportunities to complete a matrix with coloured shapes before being tested at each level. This was to ensure that the child was familiar with the concepts and task involved so that what was subsequently tested was the level of capacity to process information.

RESULTS

Regression analyses were computed with the four measures of information processing, singly, in combination and also with age controlled, as predictors of success on each of the length and number concepts. The mean STSS on the Cucui test (Mr. Peanut in this study) the earliest age of success and the percentage of the sample who succeeded on each of the length and number tasks were also calculated. Tables for each of these, and the 1987 analyses will be presented and discussed at the conference.

In summary the 1985, 1986 results indicated that the Playing Card Relations test at level 2 was the best single predictor of the following length and number tasks that required binary relations (number comparisons), binary operations (addition) and transitive inference (number seriation, length seriation and co-ordination of the size and length of units). The Matrix Completion task was equal or best as a predictor for three of the tasks (subitizing, number comparisons and subtraction) and was almost as effective as Playing Card Relations as a predictor for addition and length seriation.

Regression analyses for all four measures of capacity, in combination, showed that Playing Card Relations followed by the Matrix Completion task made the greatest contribution as predictors of performance on length and number concepts. The Family Relations test made the greatest contribution as a predictor of performance on number inferences. Further regression analyses were computed to assess the unique contribution made by each predictor variable. The matrix task made a unique contribution as a predictor variable on more tasks than any of the other measures.

On the basis of these analyses the Matrix Completion and the Playing Card Relations tasks were the strongest predictors of success with length and number tasks. Integration of number and length of units was the only task for which age as a factor increased the variance accounted for by the capacity measures.

The mean age of the sample was 6.4 years and the mean STSS was 2. This STSS was expected on the basis of hypothesised and empirical scores proposed by Case (1985:324). The percentages of the sample who succeeded at Level 2 of the Matrix, Playing Card Relations and Family Structure tasks were 0.34, 0.64 and 0.64 respectively. It would be expected on the basis of results obtained by Halford (1980) that more children by age 5 or 6 onwards would perform at Level 2 on the Matrix task. However, success at Level 2 on the capacity measures that were designed specifically for this study was as expected for children in this age range.

The earliest ages and percentages for success on length and number tasks were comparable with predictions from the literature for each task. The only result that was surprising was that for subitizing. Perhaps children at Cherbourg do not talk about "threes" and "fours" at home as much as other children. It is possible that the first real number quantifying activities for these children occur at school and depend on counting.

DISCUSSION

The results indicated that this sample of Cherbourg Aboriginal children possess capacity to process information, as measured by the Cucui and Matrix tasks that is compaorable with non-Aboriginal children at the same age. Moreover on tasks specifically designed to measure levels of thinking of concepts based on familiar knowledge systems they performed as one would expect of other children of the same age with equally familiar material.

Capacity measures were significant predictors of performance on basic mathematical concepts for length and number. In addition performance with age on the length and number tasks was comparable with results obtained in other studies. These children are cognizing concepts basic to length and number learning. If their cognitive style (cf. Watts, 1976), strategies (cf., Kearins, 1976) or processing modes (cf., Klich and Davidson, 1984) are different they are apparently nevertheless effective. It is known that the children at Cherbourg at a later stage in their schooling do not achieve as well in mathematics as their non-Aboriginal peers. The results of this study show that the lack of achievement cannot be attributed to lack of capacity or inability to cognize basic mathematical concepts. Lower achievement by Aboriginal children in school mathematics is probably a function of educational and environmental factors. Learning school mathematics may also be affected by the fact that there are variations from Standard English in the language spoken by children at Cherbourg. Some of these variations cause the children to talk about and probably cognize mathematics concepts inaccurately.

Employment opportunities at Cherbourg are limited. In order to gain employment outside the community children must succeed in school. The challenge that faces teachers of mathematics is great. School mathematics must appear to have real world meaning and language differences must be dealt with explicitly. In addition curriculum content should be analysed for demand on capacity and sequenced accordingly.

REFERENCES

Berndt, R.M. (1977). Aborigines and Change: Australia in the '70's. New Jersey: Humanities Press.

Boulton-Lewis, G.M. (1983). Length Measuring Knowledge, Information Processing Capacity in Young Children and Implications for Curriculum Planning. Doctoral Thesis, University of Queensland.

Boulton-Lewis, G.M., Neill, H. & Halford, G.S. (1986). Information processing and scholastic achievement in Aboriginal Australian children in south-east Queensland. The Aboriginal Child at School, 14 (5), 42-55.

Boulton-Lewis, G.M., (1987a). Recent cognitive theories applied to sequential length measuring knowledge in young children. British Journal of Educational Psychology, 57, 330-342.

Boulton-Lewis, G.M., Neill, H. and Halford, G.S. (1987b). Information processing and mathematical knowledge in Aboriginal Australian children in south-east Queensland. Australian Aboriginal Studies, 2, 63-65.

Boulton-Lewis, G.M., Neill, H. and Halford, G.S. (1987c). Research into Educational Potential of Aboriginal Children in South East Queensland. Report to the Australian Institute of Aboriginal Studies.

Boulton-Lewis, G.M., Neill, H. and Halford, G.S. (paper submitted). Capacity of Aboriginal children in south-east Queensland to process information and potential to learn mathematics. Journal of Mathematical Behaviour.

Bourke, S.F. and Parkin, B. (1977). The Performance of Aboriginal students. In S.F. Bourke & J.P. Keeves (Eds.), The Mastery of Literacy and Numeracy. Melbourne: Australian Council of Educational Research.

Case, R. (1985). Intellectual Development: Birth to Adulthood. New York: Academic Press.

Fischer, K.W. (1980). A theory of cognitive development. Psychological Review, 87, (6), 477-531.

Halford, G.S. (1980). A learning set approach to matrix multiple classification: evidence for a theory of cognitive levels. International Journal of Behavioural Development, 3, 409-422.

Halford, G.S. (1982). The Development of Thought. Hillsdale, N.J.: Erlbaum.

Kearins, J. (1976). Skills of desert Aboriginal children. In Kearney, G.E. and McElwain, D.W. (Eds.) (1976) below.

Kearney, G.E. and McElwain, D.W. (Eds.) (1976). Aboriginal Cognition: Retrospect and Prospect. New Jersey, U.S.A.: Humanities Press.

Klich, L.Z. and Davidson, G.R. (1984). Towards a recognition of Australian Aboriginal competence in cognitive functions. In J.R. Kirby (Ed.). Cognitive Strategies and Educational Performance. London: Academic Press.

Koepping, K. Cultural patterns on an Aboriginal settlement in Queensland. In Berndt, R.M. (1977) above.

Laboratory of Comparative Human Cognition. (1983). Culture and Cognitive development. In P.H. Mussen (Ed.) Handbook of Child Psychology. 4th edn. Vol. 1, New York: Wiley.

McElwain, D.W. and Kearney, G.E. (1973). Intellectual development. In Kearney, G.E., de Lacey, P.R. and Davidson, G.R. (Eds.) (1973). The Psychology of Aboriginal Australians. Sydney: Wiley.

Seagrim, G. and Lendon, R. (1980). Furnishing the Mind: A Comparative Study of Cognitive Development in Central Australian Aborigines. Sydney: Academic Press.

Watts, B.H. (1976). Access to Education: An Evaluation of the Aboriginal Secondary Grants Scheme. Canberra: Australian Government Publishing Service.

MATHEMATICAL VULNERABILITY

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Autobiographical in form and theory, this paper relates views of mathematics of pre-service elementary education majors to the historical development of mathematics. That is, their growth, similar to the history of mathematics, has been marked by the themes of certainty, uncertainty, and vulnerability. Central to this paper is the effort to discuss mathematics (usually a cognitive subject) and vulnerability (usually an emotional subject) in a way that does not perpetuate the separation between the two.

In one view of its historical development, mathematics has shifted away from the certainty of Euclidean geometry towards the uncertainty proved by Godel in 1930. This mathematical uncertainty has been marked by vulnerability, thus leaving mathematics open to attack. In another view, particularly that of Imre Lakatos, mathematics has always been uncertain, vulnerable, and has grown through such attack.

In this paper, these views of mathematics will be related to what I have noticed in students studying to be elementary school teachers. That is, certainty and uncertainty have been manifested in their process of questioning, reflecting, and examining their views of mathematics, a process resulting in enriched and expanded views. Investigating this process is important. If students studying to be teachers can expand and enrich their views of mathematics, then the way they teach mathematics will be expanded and enriched.

Also important to this paper is the autobiographical form it will take. Since fall, 1986, I have kept fieldnotes, journals (both personal and professional), and other important data related to my teaching of university elementary mathematics methods courses. My interest in doing such research stems from the realization that I think very differently about mathematics and its teaching since I was a junior high school teacher. My views about mathematics and mathematics teaching have also expanded and this process will be an integral part of this paper.

Therefore the theoretical basis lies in the assumption that coming to know one's self, (exploring one's own behavior), is a way of understanding the external world (exploring other peoples' behaviour). In fact, this self knowledge includes understanding one's observations of others, since one cannot separate self-understanding from all understanding.[1] This last idea, especially concerning separation, is important because the mathematical vulnerability theme is one that brings together two themes (one usually thought of as cognitive and one usually thought of as emotional) commonly studied separately.

Understanding my use of the concept of separation is crucial to the theoretical basis of this paper. That is, I can make distinctions between emotion and cognition but this is not the same as saying that they are separate. When they are viewed as separate, an artificiality is created. The difference between distinction and separation needs more discussion.

Where do I begin? With a conference in June 1988 sponsored by the School of Education and the Department of Women's Studies of the University of Haifa. The theme of the conference is "Private Women - Public Work". My talk is titled, "On Being a Nurturing Mathematics Educator: Connecting Professional and Private Lives".

When I was planning my paper, the theme struck a chord. My professional work in autobiography (in terms of studying my university teaching) was also my private life. In fact, using the word "connecting" in the title of the talk was misleading. It implies that there are two separate worlds to be connected, an implication that didn't bother me until discussing the paper with a colleague.[2] He mentioned that I might say that my mood will color how I learn or teach mathematics. But there is no uncolored knowledge. (See Heidegger, 1962.) So instead, I need to say that my mood cannot be separated from how I learn or teach mathematics, although they can be distinguished. This difference (between separating and distinguishing) is important in that it is often assumed that if I can distinguish, then those discernible things are also separate. But in the process of separating we often forget that understanding is holistic in nature. Categories like emotion and cognition (or professional and private) are helpful in making distinctions as long as we do not also assume that they are separate.

When I used the word "connecting" in the title of my Israel talk, I was concerned about making a connection that was already made. Thinking about professional and private as separate was artificial. These same points relate to the interest I have had in the interplay between emotion and cognition. Since finishing my dissertation (Brandau, 1985a), I have been struggling to find a way to study that "interplay" (e.g. Brandau, 1985b). I have also watched, with excitement, while other researchers (McLeod, 1987, for example) were doing work involving the same theme. But perhaps my thinking about emotion and cognition as separate has been underlying my struggle. As the point was made about professional and private lives, emotion and cognition are not separate at all. When you study one, you also study the other. We can make distinctions between emotion and cognition but when we make the assumption that they are also separate, we create artificiality.

There is also an important connection here to mathematics teaching and learning. If we want students to see that separating mathematics into discrete pieces is artificial, then we need to recognize that other separations, particularly ones made in research, are also artificial. We may need to make distinctions between the topics of geometry and algebra, but when we also separate them, and continue to separate them, students come to believe they are separate. Similarly, in research, when distinctions between emotion and cognition evolve into assumptions of separation, then they are studied as separate, are thought of as separate, and continue to be studied as separate.

I have been stressing the artificiality of separation because of the theme of this paper, mathematical vulnerability. Even though we may need to distinguish between the words mathematics and vulnerability, to think of them as connoting separate images (cognitive and emotional ones) is artificial. So that the impression of separateness is not perpetuated, a form of doing and reporting research is needed. I hope that this paper can move towards accomplishing this.

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A television program on NOVA titled "A Mathematical Tour" dealt with the theme of mathematics developing from a field concerned with certainty to one recognizing uncertainty. When non-Euclidean geometries were

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created, the certain world of mathematics was shattered. Euclid's "truths" were seen as self-evident and went unchallenged until the mid or late nineteenth century (Davis and Hersh, 1980). Since Euclidean geometry was seen as the foundation of a certain mathematics, when that went challenged, all of mathematics went challenged. Was mathematics now resting on a foundation of quicksand?

Set theory became the new foundation with Bertrand Russell spending much of his life trying to reformulate set theory into certainty. He wrote, "I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere."(quoted in Davis and Hersh, 1980, p.333)

The search for certainty continued until 1930 when Godel showed, with his incompleteness theorems, that certainty was impossible to achieve.

A way of dealing with this uncertainty was brilliantly shown by Imre Lakatos in Proofs and refutations (1976). Lakatos showed that mathematics and even the history of mathematics was fallible. He showed them to be dynamic processes, growing through the search for counterexamples to existing theories while simultaneously proving these theories. So proof occurs through a clash of views, and mathematics is vulnerable, yet growing because of that vulnerability.

In a world of shifting certainty and uncertainty, there would need to be less worry about control. In recognizing the uncertainty of mathematics, we have had to release the concern for control. We have had to accept the anxiety that can accompany uncertainty. That is, if mathematics can be uncertain, then we can never known when an "already proved" problem can be "disproved", or when a well-established theory can be challenged. What can be anxiety provoking is never knowing when our world will change.

These ideas have surfaced in university courses I have taught. In one class, when we were discussing the interrelationship between mathematics and philosophy, one student verbalized a sudden awareness with sadness. I had removed the last area in her life which she felt had certainty. To her mathematics clearly had had right and wrong answers. Another reaction, which was filled more with anger than sadness, occurred in an early childhood methods class where I gave a guest lecture. I had asked the students to do a problem for which several right answers can be obtained, depending on how you interpret the problem. Students were angry and confused. Many of them said, "if there are problems in mathematics with more than one right answer, then the next thing you'll be telling us is that 2 + 2 = 5!!" Their reaction indicated a loss of control. If mathematics is not certain, then it is chaos.

Facing this loss of control and uncertainty occurred in another methods course I teach. As part of the requirements, I have students keep a weekly journal. One purpose of the journal is to have a place to work mathematics problems, ones in Thinking mathematically by Mason, Burton, and Stacey (1985). This book promotes the learning of mathematics as a process, one similar to the one promoted by Lakatos. Problems in this book cannot be done in a few minutes. Students learn that doing mathematics is uncertain; sometimes answers to problems are not given; sometimes there may not even be an answer. Students learn through argument and counterargument, mostly with themselves. For all the students, working through this book is a very different mathematical experience, and one that leads to growth. I share some reflective thoughts from one of these students.[3]

> Thinking back to those first few lectures way back in September, ... I thought of math as a series of steps that followed one after the other. If the steps were taught well, math was easy. If a teacher skipped some steps than math was hard. I had a very narrow idea about math and my own personal fear further restricted that view. I always felt that a person could either do math well or couldn't do it at all and that when you did math it was either right or wrong. This course certainly changed my mind!
>
> First of all I was intimidated by how "personal" you made the math. Not only did you let us do the math ourselves but you encouraged us to openly discuss feelings, and how we tackled certain problems...
>
> My first attempts at working from the book Thinking Mathematically were disastrous and frustrating. "I can't do this" was my common complaint and I began to experience again the agony of math classes. It wasn't until well into the course that I began to put one and one together... By personally attacking the problems it became clear that there were no right or wrong methods. Math was personal and I could use which ever approach suited me best. Often problems were not solved with a straightforward answer and usually involved some thinking, figuring out and reattacking the problems from a different angle...

To help our students grow, it is also important to show ourselves as vulnerable human beings. In the Lakatosian sense, show that our ideas about mathematics (and about teaching) are open to argument. Two incidents have elicited the student reaction, "it made me feel so good to see you, a mathematics professor, struggle and show us that struggle". One incident involved a student presenting a problem her grade 7 son had to do for homework. We worked it together, as a class, with me at the overhead projector. At first I tried to do the problem by working backwards, and then switched to trial-and-error. Her comment, related to my struggle, meant that she saw that I did not know how to solve the problem at first. Perhaps implicit in her comment was the thought, "I always thought professors knew everything." The other incident involved discussion of the division concept. A student tried to help me understand her interpretation of division, one which I did not understand at the time. During coffee break she explained her view to me again; this time I understood it, and shared my new knowledge with the entire class. Here, the student referring to my struggle, was referring to my willingness to acknowledge my own growth as a learner, a willingness to relinquish control of what students often see as "expert authority".

*****

The issues of control, resistance, and vulnerability are ones dominating my life these days. A letter arrived from Kathryn, a student of mine last year and now a close personal friend.[4] It was in response to my lengthy letter describing a recent conversation with a man, one who I have not known for a long time (in terms of weeks, months, or years) and yet one to whom I told a great deal about myself. My letter to her was filled with feelings of vulnerability. It asked: How could I tell these personal intimacies to someone I didn't know? Could I trust him with this information about me? And what does "really" knowing someone mean anyway? Kathryn wrote back about risk taking, vulnerability, intimacy, responsibility, freedom, resistance, and insecurity. She wrote that she admired me for the risks I was taking. I didn't feel admirable, just scared and vulnerable.

Reading Kathryn's letter made me aware of how much this vulnerability permeates my teaching. The way I teach is risky. I'm trying to create a certain atmosphere in my methods courses, one that allows for honesty, risk taking, and freedom. The atmosphere needs to be a safe one that allows students to express their fears about mathematics honestly. This makes them vulnerable, with me and with other students. And yet such expression gives them the freedom to move beyond their fears and to learn the mathematics they feel they never learned.

It is difficult to be honest, not only with others but with one's self. When I'm honest, I'm vulnerable and admitting to a side of myself that I may not like or want to know exists. And yet being truly honest and vulnerable feels like freedom, and hence growth. Recently I was honest (and hence vulnerable) with a male friend. It was difficult and yet I now feel free--free of all the bottled-up emotions that I've been afraid to acknowledge openly with myself and with him. And I feel that I have grown; growth occurs when we risk and show vulnerability.

It is important that students be aware of my growth. I have discovered that they hear my beliefs about mathematics, and assume that I have always held these beliefs. But my school training, similar to theirs, was one that emphasized mathematics as computational skills, where problems had one right answer and one right way to do them. Theories in geometry were memorized, as immutable truths discovered once, only to be regurgitated for all time. It wasn't until graduate school that I began to view mathematics differently. My training in sociology (and theories of multiple realities) led me to question the idea of one truth and led me to reading more about mathematics, especially as written by Davis and Hersh (1980, 1986). And my training in ethnography started me asking the question: Why is this occurring the way it is? Becoming aware of the research into children's strategies of thinking helped me become aware of individual differences even in university students.

In my teaching, I try to get students to think critically about the teaching and learning of mathematics. The uncertainty that accompanies such constant questioning can be frustrating. Students want a list of the ten steps to being a successful teacher. They want certainty and control. I want them to see that teaching means constant questioning and learning. I want them to see teaching as similar to the process used by Lakatos, involving uncertainty and vulnerability.

In conclusion, to grow we must take risks and place ourselves in vulnerable positions. As this has occurred in the growth of mathematics, it can occur in the teaching and learning of mathematics and in research. But to do so, we must admit to the artificial separations of emotion, cognition, subject matter, professional, and private lives. By keeping these areas separate we are concerned with controlling them. By loosening the boundaries, perhaps even cutting them, we become open to enriching our learning about ourselves and others--about mathematics, teaching, learning, and research.

References

Brandau, L. (1985a). *One elementary teacher's struggle to encourage thinking in her mathematics classroom: a metaphorical analysis.* Unpublished doctoral dissertation, University of Illinois at Urbana-Champaign.

Brandau, L. (1985b). *Emotion and cognition---effects of fear of mathematics on learning and teaching*. Paper presented at the annual meeting of the International Group for the Psychology of Mathematics Education-- North American Chapter, October, Columbus, Ohio.

Davis, P. & Hersh, R. (1980). *The mathematical experience*. Boston: Birkhauser.

Davis, P. & Hersh, R. (1986). *Descartes' dream*. New York: Harcourt Brace Jovanovich.

Earle, W. (1972). *The autobiographical consciousness*. Chicago: Quadrangle Books.

Gadamer, H. (1977). *Philosophical hermeneutics*. Berkeley, CA: University of California Press.

Grumet, M. (1980). Autobiography and reconceptualization. *Journal of Curriculum Theorizing*, 2:2, 155-158.

Grumet, M. (1987). The politics of personal knowledge. *Curriculum Inquiry*, 17:3, 319-329.

Gunn, J. (1982). *Autobiography: towards a poetics of experience*. Philadelphia: University of Pennsylvania Press.

Heidegger, M. (1962). *Being and time*. New York: Harper and Row.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge: Cambridge University Press.

Mason, J., Burton, L., & Stacey, K. (1985). *Thinking mathematically*. Reading, MA: Addison-Wesley.

McLeod, D. (1987). Beliefs, attitudes, and emotions: affective factors in mathematics learning. *Proceedings of the Eleventh International Conference of the Psychology of Mathematics Education*, Montreal, July, 1987.

Pinar, W. (1980). Life history and educational experience. Part I *Journal of Curriculum Theorizing*, 2:2, 59-212.

Footnotes

1. It is not the purpose of this paper to explore the many issues surrounding self-understanding and autobiographical research methods. I refer the reader to the work of Earle (1972), Gadamer (1977), Grumet (1980, 1987), Gunn (1982) and Pinar (1980), to name a few.

2. Many thanks to David Jardine of The University of Calgary, whose insight, information, and inspiration is always appreciated.

3. Many thanks to Gerri Taylor for her permission to quote from her journal.

4. I owe a lot of the inspiration for my current work to Kathryn Richmond and her ideas. Thanks also go to Don Maki for his inspirational comments and discussion on a draft of this paper.

COGNITIVE PSYCHOLOGY AND MECHANISTIC VERSUS REALISTIC ARITHMETIC EDUCATION

Jan van den Brink

Summary

Cognitive development psychologists hold widely differing opinions with regard to learning addition.

According to Cobb (1987), these differences have an influence on education. We will enumerate nine controversial issues, give our standpoint on each one, and substantiate a number of these standpoints with examples taken from our own educational research.

Nine controversial issues

1. Children's ideas were not included in all theories. We feel that the theory should therefore be altered in order to include children's ideas.
2. Whether misconceptions and mistakes should be avoided? We feel that children learn from using their misconceptions as a conflict situation.
3. The influence exerted by contexts. In our opinion, 'people contexts' should be given priority over 'object contexts' in early education.
4. The role of the student - as actor or as observer - in the research. In our opinion, it is important that the students (and researchers) take on the roles of both actor and observer.
5. Whether learning is an accumulation of knowledge. We regard 'learning' rather as a growing awareness of certain actions - a reaction to conflicts with, as a result, a reorganization of existing ideas.
6. The emphasis on bare arithmetical symbols devoid of further meaning, controlled by the rules of arithmetic. We emphasize symbols which are linked to contexts and which may later form a rich bare structure (arrow-language).

7. Placing of the unchanged equal-sign between numbers and between objects. We feel that symbols of artificial languages may well be ornamented.
8. Arriving at abstractions through analogies and metaphors. We attach a great deal of importance to the referential arrow-language and to conflicts. We consider the carry-over of structures from one context to another to be characterized by:
   a. blurring of the meaning of symbols (Von Glaserfeld, Steffe & Cobb)
   b. transparency of symbols (Polanyi)
   c. flowing exchange of real-life and symbolic worlds (Freudenthal)

. The necessity of practice in order to establish the artificial similarities between bare sums and arithmetic material and in order to memorize the arithmetic facts. In our opinion, other activities are also important for practicing arithmetic facts: application, calculating through reasoning, one's own production...

Our research

We had developed a new introduction to addition and subtraction in first grade based on the so-called 'bus arrow-language'. We wanted to know whether this introduction would be 'better' than the traditional approach, keeping in mind the nine issues mentioned above.

We decided to set up a long-term (1 year) comparitive research project between the first graders at two schools: the Dreesschool (D) and the Nieuwlandschool (N). Present here were two extremely different approaches to education: realistic arithmetic education at D and traditional mechanistic education at N.

Research organization

In this research project we compared, on the one hand, the instruction as given by the two teachers at D and N and, on the other hand, the learning results of the students involved. This was carried out by means of taking regular notes on the daily instruction (for instance in a journal) and on the learning achievements (through testing the children in conversation) throughout an entire school-year.

Research object and criteria

The arrow-language was the research object whose influence on learning addition and subtraction we wished to know. In order to measure and compare learning results from both schools we chose three subjects to be used as criteria. These were tests on general arithmetic skills, the assignment of making one's own arithmetic book and tests on missing addend sums. The children at both schools were given the same assignments and tests.

At each school the children were divided into a group of test-students and a group of control-students in order to measure the influence of the testing conversations on how they learned.

Research results

We will now discuss certain results of this research against the background of some of the 9 issues.

a. Children's ideas should be included in the theory

The phenomenon that children invent divergent images and activities during arithmetic and mathematics lessons has long been confirmed by various researchers (Holt, 1980; Tall, 1980; Hart, 1981; Donaldson, 1979; Carpenter c.s., 1981). The theorists differed with regard to whether education should link up as closely as possible with the children's ideas. Through lack of a formalization of the children's ideas it was not possible to get a grasp on this . point and it was therefore necessary to avoid it in the theory forming (Anderson, 1981). Our research indicated, however, that children's ideas can be used in early education in a productive and quick fashion way when,e.g. introducing addition and subtraction. Whereas N first prac-

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ticed addition for seven weeks before introducing subtraction, both addition and subtraction were directly introduced and understood at D.

b. People, animal and toy contexts should be given priority over object contexts.
The children's research *results* showed that the mental activities of addition and subtraction could be introduced much more quickly in *contexts* in which children could *play* rather than in purely illustrative *object contexts*. People and animal contexts were evidently of much more significance to children. Later in the year it became apparent, too, that the rate of success in solving a particular type of problem depended upon the context in which the problem was placed. Difficult sums of the type: $?-b=c$ and $c=a+?$ were solved by all children when described in bus arrow-language. Solutions of one and the same bare arrow-sum varied greatly and depended upon the *type* of context under consideration (people, animals, toys, etc.).

c. Actor's role and observer's role should alternate
The combination of the roles of actor and observer was realized in the *arithmetic play-acting*. Our results indicated how suitable this manner of working indeed was:

- the children remembered the play-acting contexts accurately for a long time (3 to 5 months); these contexts therefore formed a strong basis for later arithmetic applications.
- one of the productive hints which could be given when the children made a mistake in the arrow-sums was to suggest that they tell the story step by step.

Here one can see, too, the importance of play-acting.
It would appear that the permanent role-exchange between actor and observer, which is characteristic of arithmetic play-acting, provided extremely suitable circumstances for learning arithmetic.

d. Arrow-language: the symbols are not sacred
Many researchers presented arithmetic languages to children in a simple fashion. The *use* of diagrams and notation was of primary importance, not the designing of them (Gagné, Dienes, Gal'perin, Resnick). The mathematical symbols

(+,-,=) were not 'embellished', but rather placed unaltered between numbers and between objects. On the other hand, research was conducted to see whether children could be induced to develop their own arithmetic language by, for example, making drawings of situations. Aside from the value which the illustrations had for each individual student, the researchers were disappointed by the results: the children spent a great deal of time and effort on unimportant aspects and the illustrations were not suited for general use.

In between these two extremes is the arrow-language which we have chosen.

Our research produced the following:

- an arrow was the pre-eminent symbol for evoking in young children the image of a directed movement.
- embellishment of bare arrows contributed to bestowing a variety of significances to one and the same chain of arrows.
- 3 to 5 months after the bus-play, arrow-chains still brought to mind the bus-context.
- arrow-language directed the attention and called the children **attention to** important relations within a story problem, so that they were then able to solve the sum.
- arrow-language was used as a subsidiary to the bus during registration of class bus rides.
- by altering the embellishment of the arrows, the language assisted the children in dealing with all sorts of significances. Arrow-language was used as an intermediary for moving from one context to another.
- the arrows themselves were used as embellishment in bare **equal-sums**, by which means these sums acquired an explanatory significance.
- finally, the arrows separated from the significance. The original meaning began to blur. This was evident from the fact that children developed entirely new symbol combinations.

symbols in arithmetic languages (+,-,=) should not be regarded as sacred cows. Embellishment of the arrows and use of the arrows as embellishments in other languages are meaningful activities. The function of arrow-language when dealing more abstractly must also be mentioned here. The transition from context to context is possible via arrow-language, as became apparent from the following phenomena:

- when given the assignment of inventing as many different arrow-sums as possible, the children designed examples both from real life and from the bare world of arrows (Freudenthal).
- Addition of contextual embellishment made the arithmetic language more transparent: the meaning was recognized (Polyani).
- The original significance became blurred when children invented their own symbol combinations (Von Glaserfeld, Steffe, Cobb).

These were the three characteristics of the carry-over of structures from one context to another: flowing exchange of real-life and symbolic worlds, transparency of symbols and blurring of the original significance.

In conclusion

The opinions of cognitive development psychologists, which differed greatly from other, formed the background for each of the extreme educational forms. Each form of education led to varying learning achievements.

Literatuur

[1] Anderson, J.R. (ed.) *Cognitive skills and their acquisition.* Hillsdale N.J., Erlbaum, 1981
[2] Brink, F.J. van den *Mutual observation.* In: For the learning of mathematics, 2(1981)Nov., pp.29-30.
[3] Carpenter, T.P., J. Hiebert & J.M. Moser. *Problem structure and first-grade children's initial solution processes for simple addition and subtraction problems.* In: Journal for research in mathematics education, 12(1981)nr. 1, pp.27-39.
[4] Cobb, P. *Information processing psychology and mathematics education: a constructivist perspective.* In: Journal of mathematical behavior, 6(1987) pp. 3-40.
[5] Donaldson, M. *Children's minds.* Glasgow, Collins, 1979.
[6] Freudenthal, H. *Theorievorming bij het wiskundeonderwijs, geraamte en gereedschap.* In: Panamapost, 5(1987)nr.3, pp. 4-15.
[7] Glasersfeld, E. von & P. Cobb. *Knowledge as environmental fit.* In: Man-environment systems, 13(1983)nr. 5, Sept.
[8] Glasersfeld, E. von & L.P. Steffe. *Composite Units and the Operations that constitute them.* In: Proceedings of the tenth International Conference Psych. of Math. Education PME 10, 1986 (July), London, University of London, pp. 212-216.
[9] Hart, K. *Children's understanding of mathematics: 11-16.* London, John Murray, 1981
[10] Holt, J. *How children fail.* New York, Pitman, 1980.
[11] Polanyi, M. *Personal knowledge.* Chicago, Univ. of Chicago Press, 1962.
[12] Tall, D. *Mathematical intuition, with special reference to limiting processes.* In: Proceedings of the 4th International conference for the PME, 1980, aug., Berkeley, pp.170-175

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PROOF AND MEASUREMENT: AN UNEXPECTED MISCONCEPTION [1]

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This article begins by providing background information about ongoing research into students' understandings of the differences between measurement of examples and deductive proof. This research uses the microcomputer programs, the GEOMETRIC SUPPOSERS, environments where both measurement and proof play important roles. The article then goes on to describe an unanticipated misconception about these two methods of verifying statements. This misconception is discussed as it arises in the work on one high school student.

INTRODUCTION

The GEOMETRIC SUPPOSER software environment is a series of microcomputer programs which allows students to create diagrams, explore them using measurement, make conjectures, and test these conjectures empirically before beginning to prove them deductively. Learning Euclidean geometry with the GEOMETRIC SUPPOSERS forces students to think about proof differently than they would in a traditional course. Students confront the issue of the relationship between measurement and proof. From a mathematician's perspective, it is very important that students understand that examples with confirming measurements do not a proof make. For example, to prove that "An altitude from the vertex angle in an isosceles triangle divides the triangle into two triangles of equal area.", it is not sufficient to show three examples with measurements which support this statement.

One way to emphasize the difference between measurement and deductive proof is to highlight two important characteristics of measurement. With measurement, one can only check a finite number of cases with limited precision. If a statement has a universal quantifier and is about an infinite number of objects (e.g. all isosceles triangles), then by using measurement one can never be certain that all examples will support the statement. Maybe there exists a set of cases for which the conclusion is not true though the premise is satisfied. One can never check all cases.

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[1] The proofs discussed in this paper are typical high school Euclidean geometry proofs. They do not include proof by construction or proof by mathematical induction.

Second, measurement involves tools that by definition must have a margin of error. All statements based on measurement must be qualified (implicitly or explicitly) by the limitations of the tool. Thus, in measuring lengths with the GEOMETRIC SUPPOSERS, one can say that the two measurements are equal to one one-hundredth of a length unit. The segments may not be equal if measured with a more precise instrument. Thus, even a statement of equality which holds for only one triangle cannot be proven, in a mathematical sense, by measurement.

These characteristics of measurement are complicated by the role which counterexamples play in mathematics. While measurement cannot prove an universal assertion about an infinite set of objects, it can disprove such an assertion. Sufficiently large measurement discrepancies (beyond the precision and accuracy limitations of the tool) which contradict a statement which begins "For all triangles,..." prove that the statement is false. Thus, there is a lack of symmetry between the power of measurement to disprove universal statements about infinite sets and its lack of power to prove such statements.

Deductive proof has three important characteristics which help distinguish it from empirical reasoning. First, deductive reasoning guarantees that its conclusions are true for all members of the given set, even if that set includes an infinite number of elements. Second, if the results of a deductive argument indicate that two quantities are equal, then these quantities are exactly equal no matter what scale is used. Third, deductive arguments can also provide an element of illumination, or insight into why the statement is true (see Bell, 1976).

An initial concern expressed about teaching students with the GEOMETRIC SUPPOSERS was that students would not appreciate the differences between measurement and proof, that they would treat measurement as mathematical proof. These concerns led us to focus on GEOMETRIC SUPPOSERS students' ability to write proofs (Yerushalmy, 1986; Yerushalmy et.al., 1987). In fact, a corollary of the concern about work with the GEOMETRIC SUPPOSERS mentioned above is that students using the GEOMETRIC SUPPOSERS will no longer see the need for deductive proof and will therefore not learn how to write proofs as well as students in traditional classes. From our first two studies, this does not seem to be the case. Students in GEOMETRIC SUPPOSERS classes produced more formal proofs on posttests (see Yerushalmy, 1986 and Yerushalmy et.al, 1987). However, these results do not necessarily indicate that students understand the differences between measurement and deductive proof. Instead students may have become more interested in proof because they are invested in conjectures which they have devised on their own and therefore want to prove them. Alternatively, the desire to complete their conjectures by proving them may be motivated purely by the teachers' insistence that conjectures are not true until proven.

This article presents one story from ongoing research into students' understanding of the differences between proof and measurement as ways of ascertaining truth. It presents an unexpected misconception exhibited by a student in a typical, non-SUPPOSER classroom. Before presenting Larry's work, a quick description of the research background, the unit and tests used to evaluate the unit, will be provided.

THE UNIT

The unit on the differences between measurement of examples and deductive proof provides teachers with problems for student exploration in a computer laboratory setting, or for whole group exploration, using the GEOMETRIC SUPPOSERS. The teachers were provided with materials which explained the arguments that could be made about the differences between proof and measurement and which suggested ways to lead discussions. The provided materials were anticipated to require two weeks of classroom time. This unit was piloted during April and May of 1987 in two experimental classes. One of the experimental classes had not used the GEOMETRIC SUPPOSERS before doing the unit, one had used them all year. There were two comparison classes which were matched classes taught by the same two teachers. One of the comparison classes had not used the GEOMETRIC SUPPOSERS at all. The other used the SUPPOSERS all year and did the problems in the proof unit without discussing them.

	Used Supposer All Year	Did Not Use Supposer Prior To Unit
Experimental Classes	Did unit	Did unit
Comparison Classes	Did problems, no discussion	Never used the Supposer

THE TEST

To investigate the students understanding of proof, pre/posttests were constructed to ascertain if students felt that empirical verification constitutes a proof. The tests had three parts. Part A presented a list of statements about proof and solicited students' views asking them to agree or disagree. Part B asked students to write their own proofs. Part C of the test was patterned on work done by Martin and Harel (1986). It provided a statement with eight different arguments for one given statement. Each argument was on a different page. After reading the argument, students were asked to decide whether the argument was a convincing argument or not and whether it was a good argument or not. They were then asked to justify their opinions.

Of the eight arguments, four were inductive arguments and four were deductive ones. Below, we outline each argument.

INDUCTIVE ARGUMENTS

#1--An example for which the conditions are satisfied and the statement holds, and an example for which the conditions are not satisfied and the statement does not hold.

#3--Four examples for which the conditions are satisfied and the statement holds.

#5--An extreme or complicated example in which the statement holds.

#8--A repetition of the statement.

DEDUCTIVE ARGUMENTS

#2--A deductive proof of the statement in two columns for a subset of the described objects (a proof for a square with a diagonal, when the statement is about squares with any line through the square).

#4--A complete deductive proof of the statement in two columns.

#6--A circular deductive proof in two columns.

#7--An informal argument that is deductive and correctly done.

After giving the tests to the four classes, students' answers were classified as "correct" or "incorrect." A sample of students including students who had given each of these kinds of answers was selected for interviewing. The purpose of these interviews was to check whether students understood the questions asked in the test and to check whether students' answers had been correctly categorized.

The following story is taken from the interview of a student from the comparison class which did not use the unit and which had never used the GEOMETRIC SUPPOSERS. Initially, Larry was interviewed because his responses to the inductive arguments were "the right answers." It turned out that he had a pernicious misconception about deductive proof. He thought that deductive proof is as limited as measurement, that it can only shed light on an individual case. Not all students share Larry's misconception, but his misunderstanding emphasizes that students' understandings of similarities between measurement (scientific verification) and proof (mathematical verification) is very important, even in non-SUPPOSER classrooms.

LARRY

According to Larry's teacher, Larry is a poor student, yet a very bright boy. In his geometry class, he received a D for the year. In her words, "if he just would have applied himself, he could have done well." Her feeling was that conceptually he understood the material in the course, but that he did not take the time to practice the skills that he needed to do well in the course.

On part A of the test, Larry had answered on both pretest and posttest that he did not agree that three examples make a proof. In part C, he found none of the empirical, example-based arguments on the pre or posttest convincing, though #3 (four examples) and #5 (a complicated example) on the posttest were considered "good". Furthermore his comments on the posttest on three of the four empirical arguments seemed appropriate:

> (Examples for and against)- "This doesn't tell me what happens in every other possible case. Appearances are sometimes deceiving."
>
> (Four examples)- "These could be the only cases where the statement is true."
>
> (Complicated example)- "Might be something special about these squares to make it true here."

When interviewed, Larry's definitions of "good" and "convincing" were:

> "I thought convincing meant that it would be always true--that it [the argument] proved that the statement would always be true... [Good means that] they went about trying to prove it quite well."

Thus, it was surprising to hear him explain his written comments on the posttest deductive proofs (He had not written explanatory comments on the pretest). On the deductive proof where the line through the square was a diagonal, Larry read very carefully and saw that there was a mistake in the correspondence of the two congruent triangles on the test form. He chose "not convincing" and "not good" and wrote, "The triangles named in statement 3 are not congruent. Doesn't tell me what happens in every other case." The last sentence seemed appropriate because the proof was a proof of a special case, the line contains a diagonal. In explaining his comment Larry said, "I was just thinking of other squares." The interviewer asked, "This proof only shows you if it [the statement] is true for this particular picture?" Larry answered, "Yes, I don't know for sure about other things."

Curious about this last exchange, the interviewer went on to the other deductive arguments. For the correct, deductive argument, Larry had chosen "not convincing" and "good" and wrote, "This may apply only in this case." For the circular, flawed deductive proof, Larry made the same choices and wrote "Might just be this case." In the interview, Larry indicated that he made these choices because he felt that each argument was a good way to go about proving the statement, but he was not sure if the arguments applied only in the pictured cases or in many cases.

Larry, on his own, raised another piece of evidence that made it clear that he did not think that deductive proofs held for cases not pictured. The diagram accompanying the last argument showed three squares of equal size sharing the same center E and a line through E. The argument was that "No matter how you turn the square around the center E, the areas of the two regions formed by the cut FG will always be the same." Larry said "The last [argument] I said it was convincing, the last one.... I said it was convincing in all cases." The interviewer, "Because it says, no matter how your turn it, that it will always be true?" Larry, "It wasn't very clear." Larry chose "not good" since the argument wasn't clear and "convincing" since it took all cases into account.

One final indication that Larry thought that deductive proofs are proofs of specific cases is that at the end of part C when asked what he now knew about the statement that he didn't know before, he wrote "I think that it is probably true." According to his conception the "probably" is warranted, he has seen evidence for a number of specific cases, but no general proof.

How would Larry react if he was given a deductive proof of a statement complete with a diagram and then was given another diagram that satisfied the conditions of the statement (e.g. an isomorphic diagram with different labels)? Would he consider it necessary to write another proof with the exact same statements? The evidence from our interview suggests that he would. Unfortunately, we were not able to test out this hypothesis.

CONCLUSION

As pointed out in the introduction, one of the key differences between measurement and deductive proof is that deductive proof can prove statements which hold for infinite sets, while measurement can only verify a statement within certain bounds for a finite set. It was anticipated that some students would not understand that measurement is only effective in a finite number of cases. It was a surprise to find students who do not understand that deductive proofs hold for all diagrams which satisfy the initial conditions. Larry's views make it clear that the issue of generality and specificity is a fruitful one to study. Future research will investigate whether Larry's conception is held by many high school geometry students.

The GEOMETRIC SUPPOSERS encourage the user to construct more than one diagram for a statement. For students like Larry, the SUPPOSERS are an especially valuable tool. It would be harder for his conviction to survive a GEOMETRIC SUPPOSERS classroom than a traditional classroom where it is rare to find two different drawings for the same statement.

Work on "diagnostic teaching" (Bell, 1983; Bell, 1986) also suggests that classroom activities which challenge students' misconceptions about measurement and proof (like Larry's) and conflict-discussions which examine the similarities and differences between these two methods for verification may be effective in eradicating student misconceptions. These sorts of activities may also promote long term learning which is more successfully transfered to new mathematical domains.

BIBLIOGRAPHY

Bell, A. (1976) "A Study of Pupil's Proof Explanations in Mathematical Situations." Educational Studies in Mathematics. vol. 7. pp. 23-40.

Bell, A. (1983) "Diagnostic Teaching: The Design of Teaching Using Research on Understanding." International Reviews on Mathematical Education. Vol. 15, no.2. p. 83-89.

Bell, A. (1986) "Outcomes of the Diagnostic Teaching Project." in (ed.) L. Burton and C. Hoyles. Proceedings of the Tenth International Psychology of Mathematics Education. London. p. 331-336.

Harel, G. and G. Martin (1986) "The Concept of Proof held by Preservice Elementary Teachers." in (ed.) L. Burton and C. Hoyles. Proceedings of the Tenth International Psychology of Mathematics Education. London. p. 386-391.

Yerushalmy, M., D. Chazan, M. Gordon, R. Houde. (1987) Geometry Research Group: Technical Report. Cambridge, Harvard Graduate School of Education, Educational Technology Center.

Yerushalmy, M. (1986) Induction and Generalization: An Experiment in Teaching and Learning High School Geometry. Unpublished dissertation. Cambridge, MA. Harvard Graduate School of Education.

'DISCRETE' FRACTION CONCEPTS AND COGNITIVE STRUCTURE

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Data obtained from investigating the 'discrete' fraction concepts of 59 students in Grades 4, 5, and 6 in three Papua New Guinea Community Schools are analyzed. These data derive from three different kinds of tasks, namely 'sharing' tasks, 'discrete' tasks involving formal fraction language, and 'symbol manipulation' fraction tasks, all concerned with the fractions 1/2, 1/4, and 1/3. While all 59 students were confident and accurate when performing the 'sharing' tasks, they were much less successful on corresponding tasks in the other two categories. It is concluded that, in the teaching of fractions, reality-based 'sharing' concepts should be linked with formal language, and both of these with the symbolic manipulation of fractions. These links need to be firmly established in the learners' cognitive structures.

1. INTRODUCTION

In November and December 1987 the authors administered three paper-and-pencil tests to 283 children in Grades 4, 5, and 6 in three Community Schools in different villages in Papua New Guinea (PNG). We also interviewed, on a one-to-one basis, 59 of the children who had taken the tests, the aim being to map the children's cognitive structures with respect to fraction concepts, and especially concepts for the fractions 1/2, 1/4, and 1/3.

During the interviews children responded to a wide range of fraction tasks. This paper, however, is solely concerned with analysing data pertaining to 'discrete' fraction tasks. (An example of a 'discrete' fraction task would be to show a child 12 objects and to task him/her to pick up one-third of them; by contrast, asking a child to shade one-third of a rectangle is an example of a 'continuous' fraction task - see Clements & Del Campo, 1987.)

Hunting (1986, pp.212-213) has pointed out that traditional approaches to the teaching of initial fraction ideas have been based, almost exclusively, on the partitioning of continuous material (e.g. apples, cakes, and pies), and sections in school textbooks on fractions incorporate mostly graphic material which shows regions partitioned into various fractional units. This is probably true of the situation in Papua New Guinea (see, for instance, the section on 'Fractions' (pp.65-72) in the PNG Department of Education's (1986) *Community School Mathematics 4A).*

This, together with the fact that, while we were observing mathematics classes in the

Community Schools which we visited, we noticed that children in Grades 4 through 6 were often expected to be able to find the value of expressions such as '5/11 of 792', made us wonder whether the children's cognitive structures included the verbal knowledge, the imagery, and the memory of relevant episodes (Gagné and White, 1978) which would enable them to make sense of the symbol manipulations which they were struggling to perform. Therefore, we decided to attempt to map the cognitive structures of samples of children in the school with respect to fraction concepts, and to include within the investigation a special study of the children's responses to discrete fraction tasks.

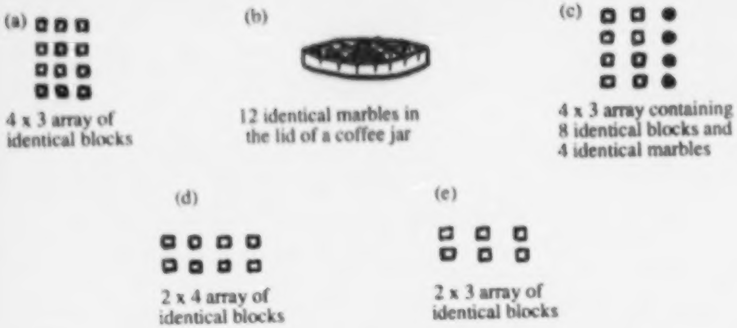
2. METHOD

Sample: 24 of the 59 interviewees constituted the only Grade 5 class at a small Community School. Nine of the 24 students were female, and 15 male. Data from the pencil-and-paper tests indicated that while these 24 students were spread across a wide spectrum of mathematical ability, on the whole the class was above average in mathematical performance for Grade 5 students in Papua New Guinea. The other 35 interviewees (20 female, 15 male) were in Grades 4, 5, and 6 (8 in Grade 4, 16 in Grade 5, and 11 in Grade 6) at two larger Community Schools (which were in the same PNG province, but in a different province from the first school). Again, while these 35 students were spread across a wide range of mathematical ability, the two schools which they attended are recognized, within PNG, as having high academic standards.

The interviews were mainly in English. If the interviewers suspected that a child was having difficulty in understanding the basic instruction for a task then an adult (usually a teacher) was called on to present the task in the child's first language. In each of the three schools English is the language of instruction.

The Discrete Fraction Tasks: The five different kinds of interview tasks which could be regarded as 'discrete' (or in the case of the Equilateral Triangle Perimeter task, having both 'discrete' and 'continuous' aspects) are now described.

1. The Array Tasks, and the Coffee Lid Task: These tasks were based in the five sets of objects shown in Figure 1. For the 4 x 3 array of blocks in Figure 6(a) interviewees were asked to give the interviewer one-half (then one-quarter, then one-third) of the blocks; similar requests were made for the objects in Figures 6(b) and 6(c); for the 2 x 4 array (Figure 6(d)) interviewees were asked to give one-half (then one-quarter) of the blocks; and for the 2 x 3 array (Figure 6(e)) they were asked to give one-third of the blocks.



4 x 3 array of identical blocks

12 identical marbles in the lid of a coffee jar

4 x 3 array containing 8 identical blocks and 4 identical marbles

2 x 4 array of identical blocks

2 x 3 array of identical blocks

Fig. 1 : The array and coffee lid task materials

2. The Cups of Water Task Interviewees were shown the three identical clear plastic containers in Figure 2(a), with one container full with water and two empty, and asked, 'What fractions of the cups have water in them?' Then they were shown the situation in Figure 2(b), and asked the same question. Depending on responses, the interviewer could ask further similar questions (eg. with three containers full with water and one empty).

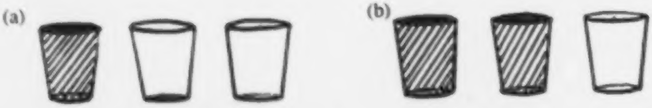


Fig. 2 : What fraction of the cups have water in them?

3. The Equilateral Triangle Perimeter Task: Interviewees were given an A4 piece of paper on which three equilateral triangles, each with vertices labelled A, B, C, were drawn (see Figure 3). They were told that Mary wanted to move around the triangle, starting at A and going through B, then through C, before arriving back at A again. While these instructions were being given the interviewer demonstrated the meaning of what was being said by pointing to a triangle. Then the interviewee was asked to indicate on the triangle where Mary would be when she had moved 1/2 (then 1/4, then 1/3) of the way around the triangle.

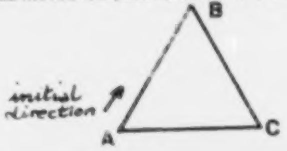


Fig. 3 : Moving around the triangle

This perimeter task has both discrete and continuous aspects : continuous, because it involves the continuous quantity, length; but discrete because the triangle has three sides.

4. The Symbol Manipulation Tasks: Interviewees were asked to work out the answers to the following three sums, which were written on a piece of paper.

$\frac{1}{4}$ of 12 = $\qquad$ $\frac{1}{3} \times 12 =$ $\qquad$ $\frac{3}{4} \times 12 =$

5. The Discrete Sharing Task: Interviewees were given 12 marbles and were shown four stick-figure pictures of 'friends'. They were asked to share the marbles amoung the four friends so that each got the same number of marbles. The task was then repeated with two (then three) stick-figure pictures of friends.

3. RESULTS

1. The Array and Coffee Lid Tasks: Table 1 shows the percentages of interviewees giving correct responses on the five tasks, for each of the fractions 1/2, 1/4 and 1/3.

Table 1 Percentages of Correct Responses on 'Expressive Discrete' Array and Coffee Lid Tasks (*n = 59 , children in Grades 4 through 6)*

	Fraction		
Task	**1/2**	**1/4**	**1/3**
1. 4 x 3 array (12 blocks)	44%	17%	29%
2. 12 marbles in coffee lid	41%	10%	17%
3. 4 x 3 array (8 blocks, 4 marbles)	39%	12%	17%
4. 2 x 4 array	41%	25%	-*
5. 2 x 3 array	-*	-*	11%

*means the task did not apply to the fraction

It can be seen that most interviewees did not demonstrate an expressive understanding of the request to give the interviewer 1/2 (or 1/4, or 1/3) of a small set of objects. The most common error was for students to give 2, 4, and 3 blocks in response to the request for 1/2, 1/4, and 1/3 of the blocks, respectively. In fact, 25% of all requests for 1/2 of a set of blocks yielded the response '2'; 62% of all requests for 1/4 of a set of blocks yielded the response '4'; and

65% of all requests for 1/3 of a set of blocks produced the response '3'. The concepts of 1/2, 1/4, and 1/3 are inexorably linked with the numbers 2, 4 and 3, respectively, in many children's cognitive structures.

2. The 'Cups of Water' Tasks: Table 2 shows the percentages of interviewees giving correct responses to the two set tasks in this category. Interviews suggested that children had difficulty trying to reconcile the displays of cups and water with their concepts of fractions. For most interviewees this was a novel, and confusing idea. Many students just guessed an answer. They were asked to name a fraction so they said '1/4', or '1/2', or whatever first came into their head.

Table 2 Percentages of Correct Responses on the two 'Cups of Water' Tasks *(n=59)*

Task	Percentage Correct	Most Common Error(s)
1. One cup of water, and two empty cups: "What fraction of the cups have water in them?" (Correct response: 1/3)	33%	1/4 (37%) of interviewees gave this response)
2. Two cups of water, and one empty cup: "What fraction of the cups have water in them?" (Correct response : 2/3)	23%	1/3 (20% of interviewees gave this response) 1/2 (13%) 1/4 (13%)

3. The Equilateral Triangle Perimeter Task: The percentages of correct responses on this task for the fractions 1/2, 1/4, and 1/3 were 14%, 5%, and 7%, respectively. Even when correct responses were given, in almost all cases they were obviously guesses. The cognitive structures of the interviewees clearly did not link the perimeter of a triangle with fractions. One might have reasonably expected that on this task a fairly high percentage of correct responses would have been given for the '1/3' task, but this expectation was not realized.

4. The Symbol Manipulation Tasks: Table 3 shows the percentages of correct responses for the three tasks in this category, and the most common errors on the tasks. Written transcripts produced by the interviewees on these tasks often revealed a desire to 'cancel'. Usually, however, what was done suggested a serious lack of understanding. Thus, for example, a Grade 5 boy wrote:

$$\frac{1}{\cancel{4}} \text{ of } \cancel{12}^{16} = 16 \qquad \frac{1}{\cancel{3}} \times \cancel{12}^{15} = 15 \qquad \cancel{\frac{1}{2}} \times \cancel{12}^{14} = 14$$

This boy knew he had to cancel, but he had no idea of what cancelling meant.

Table 3 Performances on Symbol Manipulation Tasks (n=59)

Task	Percentage Correct	Most Common Errors
$\frac{1}{4}$ of 12 =	51%	'6' (7 students gave this response) '8' (5 students gave this response) '4' (3 students gave this response)
$\frac{1}{3}$ x 12 =	48%	'3' (5 students gave this response)
$\frac{3}{4}$ x 12 =	20%	'3' (5 students gave this response) '48' (4 students gave this response)

5. The Discrete Sharing Tasks: When asked to share 12 marbles equally among four 'stick-figure' friends all 59 interviewees quickly gave the correct answer. Thirty used a 'one-for-one' procedure, building up piles of marbles on the pictures of the friends, and the other 29 immediately picked up three marbles for each friend. No mistakes were made on the other two sharing tasks (involving sharing 12 marbles among 3, then 2, 'stick-figure' friends). Usually, if an interviewee used a 'one-for-one' procedure for the first sharing task, among four friends, then he/she used the same procedure for the other two sharing tasks.

4. DISCUSSION

The preceding data analyses indicate that the reality-based sharing concepts which all the interviewees possessed were rarely linked in the children's minds with the formal language of fractions or with the symbolic manipulation of fractions; also, formal language of fractions was not linked with symbol manipulation. Students in Grades 4, 5, and 6 were spending large amounts of classroom time working with fraction symbols but did not link what they were doing with reality or the formal language of fractions.

This realization prompted us to carry out the analysis shown in Table 4. This Table shows, for both 1/4 and 1/3, those who (1) correctly performed symbol manipulation tasks and both the first 4x3 (with 12 identical blocks) array task and the Coffee-Lid task; or (2) correctly performed one, but not both of the symbol manipulation and the two 12-object tasks; or (3) performed neither the symbol manipulation nor the two 12-object tasks correctly.

Clearly, from Table 4, hardly any of the 59 interviewees associated the symbol manipulation tasks (which were much easier than those they were being asked to do in class) with tasks which required them to identify 1/4 (or 1/3) of a set of 12 objects.

Table 4 Performance on Symbol Manipulation and Two 12-Object Tasks (n=59)

Comparison 1/4	Number in Category	Comparison 1/3	Number in Category
' $\frac{1}{4}$ of 12 = ' correct, and both corresponding 12- object tasks correct.	2	' x 12 = ' correct, and both corresponding 12- object tasks correct.	1
' $\frac{1}{4}$ of 12 = ' correct, but either one or both of the corresponding 12- object tasks incorrect.	27	' $\frac{1}{3}$ x 12 = ' correct, but either one or both of the corresponding 12- object tasks incorrect.	27
' $\frac{1}{4}$ of 12 = ' incorrect, but both corresponding 12-object tasks correct.	3	' $\frac{1}{3}$ x 12 = ' incorrect, but both corresponding 12-object tasks correct.	8
' $\frac{1}{4}$ of 12 = ' incorrect, and either one or both of the corresponding 12-object tasks incorrect.	27	' $\frac{1}{3}$ x12 = ' incorrect, and either one or both of the corresponding 12-object tasks incorrect.	23
TOTAL	59	TOTAL	59

Almost one-half of the interviewees correctly stated the values of ' $\frac{1}{4}$ of 12= ' and ' $\frac{1}{3}$x 12= ' yet did not respond correctly to the corresponding array and coffee lid tasks. A few interviewees performed the latter 12-object tasks correctly, but gave incorrect answers to the symbol manipulation tasks. Many could do neither. Significantly, all had no trouble sharing 12 objects equally among 4 (or 3) 'friends'.

We believe the implications of our analysis represent a powerful indictment of prevailing practice in the teaching of fractions. And we would be more than naive if we thought that what we have found in three PNG Community Schools is not more or less true around the world, wherever schooling occurs.

Teachers, textbook writers, and mathematics curriculum developers must plan programs which link familiar real-world concepts (e.g. 'sharing'), with corresponding formal mathematical language (e.g. 'one-quarter of'), and with symbol manipulation (e.g. ' of 12 = '). This is illustrated in Figure 4. We would comment, too, that the ideas implicit in Figure 4 certainly do not apply only to fraction concepts (see Lean, Clements, & Del Campo, in press, where the ideas are applied to arithmetic word problems).

Providing classroom experiences which assist students to make the cognitive links indicated in Figure 4 requires some understanding of children's minds, and knowledge of the effects on the children of previous learning experiences.

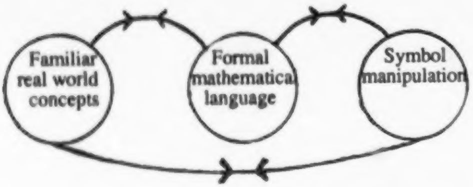


Fig.4 : Establishing links in cognitive structure

While in this paper we have concentrated on the 59 interviewees' knowledge of **discrete** fraction concepts, in fact our data base contains more information on their **continuous** fraction concepts than on their discrete fraction concepts. We are currently preparing a larger report covering all the data : however, we should say, here, that our overall picture indicates that, as a result of their schooling, the 59 interviewees did not associate formal fraction language with discrete sets of objects. This became clear when we asked them to draw a picture of 'one-half' (then 'one-quarter', then 'one-third'). Fifty-eight of the 59 students chose to represent each fraction as a sub-area of a circle or a square - the other student chose an eqilateral triangle. The idea of representing a fraction as a subset of a larger set of objects apparently did not occur to anyone. This is the kind of knowledge which program developers, and teachers, need to have.

REFERENCES

Clements, M.A., & Del Campo, G. (1987). Fractional understanding of fractions: Variations in children's understanding of fractional concepts, across embodiments (Grades 2 through 5). In *Proceedings of the Second International Seminar on Misconceptions and Educational Strategies in Science and Mathematics*. New York: Cornell University, Vol. 3, 98-110.

Gagné, R.M., & White, R.T. (1978). Memory structures and learning outcomes. *Review of Educational Research*, *48*, 187-222.

Hunting, R.P. (1986). Strategies for teaching fractions effectively: What to use. In N.F. Ellerton, (Ed.), *Mathematics: Who needs what?* Melbourne: Mathematical Association of Victoria, 213-217.

Lean, G.A., Clements, M.A., & Del Campo, G. (in press). Linguistic and pedagogical factors affecting children's understanding of arithmetic word problems: A comparative study. *Educational Studies in Mathematics.*

Papua New Guinea Department of Education (1986). *Community School mathematics 4A*. Sydney: Macmillan.

ALGORITHMIC THINKING OF DEAF PUPILS - REPORT ON A THREE YEARS CURRICULUM PROJECT

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Abstract:

In the following we report on the curriculum project "Algorithmisches Denken im Mathematikunterricht mit Hörgeschädigten" (algorithmic thinking in mathematics lessons of deaf pupils), which has been run under our scientific supervision by the ministry of education and the ministry of social affairs in our state. Divergent from the usual role of research we were in charge of the inventing of didactical situations and teaching methodology. In a smaller amount we made some investigations on the pupil's process of algorithmic concept formation. We did this in a close cooperation with "Forschungsinstitut für Mathematikdidaktik e.V. Osnabrück". We are obliged to this research institute for mathematics education for an importand progress in the theoretical framework (Schwank 1986).

INTRODUCTION

The curriculum project with deaf pupils, which we are presenting in the following paper, has two roots: one in mathematics education and one in cognitive science.

In mathematics education we have a long tradition in developing materials and lesson courses by which even primary pupils get an insight into the understanding of fundamental ideas concerning automaticaly running processes and programming computers.

In cognitive science we had a hypothesis on the value of nonverbal forms of representing mathematical concepts arrising from the theoretical and experimental analysis of pupils' concept formation processes. This led us to the idea of testing the theory with deaf pupils as a kind of crucial experiment.

Concerning the research on deaf pupils one may find in cognitive science some research on language acquisition of deaf but not on their mathematical concept formation processes. Similarly, there is no research reported in the field of mathematics education concerning the deaf, as far as it can be seen for example in the abstract book of the International Congress on Education of the Deaf held in Hamburg 1980 or in the recent volumes of international journals in mathematics education. Therefore the curriculum project on which we are reporting is an exception (Cohors-Fresenborg 1987b).

Since 1979 there has been done research concerning the question of algorithmic concept formation of deaf pupils by the group 'foundations of mathematics and mathematics education' at the university of Osnabrück. The investigations with primary pupils led to the result that deaf pupils are able to construct and analyse algorithms using the didactical material'dynamic mazes' (which is explained below) on a considerably high level compared with healthy pupils of the same age. The investigations were extended to the field of the fundamentals of computer programming using the model computer registermachine. As a consequence of this successful work the ministry of science of Lower Saxony has supported a pilot study from 1981 to 1983. One result was the hypothesis that the remarkable high level of the deaf pupils in constructing and analyzing automata networks was based on their experience in the organizing of actions, so that by a consequent developement of those abilities there could be an intuitive basis for an understanding of fundamental mathematical ideas (Cohors-Fresenborg/Strüber 1982).

HYPOTHESIS

The following hypothesis was presented as the basis of a three-years-curriculum project:

By changing the philosophy of mathematics lessons using the intuitive basis of algorithms it is possible, that deaf pupils are able to understand complex mathematical concepts from which they are excluded by the kind of mathematics curriulum which is used today.

We were convinced that the reason for this can be seen in the fact that the philosophy of pure mathematics is - from a philosophical point of view - based on a language-orientated way of mathematical concept formation.
Our curriculum project should show that it is possible to teach the usual contents of school mathematics (algebra and functions) using algorithms as a fundamental idea so that deaf pupils can reach a performance which is comparable with that of healthy pupils in a middle-range of ability in secondary schools.
Using the deaf pupils' understanding of fundamental algorithmic concepts it should be possible to teach programming of a high level language like PASCAL. This should open to them better chances in finding more qualified jobs than they could find up to now.

The center of our research was the constructing of curriculum elements using the idea of algorithms. We were convinced that we could only reach our high level aims, if we gave preference to the constructing of suitable mental models (Johnson-Laird 1983) for algorithms, and not by teaching mathematical facts.

CURRICULUM PROJECT

From 1984 - 1987 there was made a curriculum project in the center for the education of deaf pupils in Osnabrück. Our job was the scientific consulting. We understood our job as the task to develop curriculum elements and the suitable teaching-methodology.
During our project there have been taught two classes:
One class beginning with grade 8 (up to the end of grade 10), and one class beginning with grade 6 (up to the end of grade 8).

Didactical material

The use of the material 'Dynamische Labyrinthe' (Dynamic Mazes) and the model-computer 'Registermachine' played an important role . Both of them have been used before in other curriculum projects in primary and

secondary schools, so that there existed quite a lot of experiences concerning the healthy pupils.
In 1974 we started to develop a didactical material "Dynamische Labyrinthe" (Dynamic Mazes) with the aim to give pupils even at primary level an understanding of the fundamental ideas which are necessary to understand automaticaly running processes and programming computers. The mathematical analysis of toy railway networks, in which only one train may run which itself changes all the points automatically on his run, leads to the idea that such a railway network may be regarded as a sequentially running network of simple automata. The points are regarded as a finite automaton with two states (left and right). The box "Dynamische Labyrinthe" contains such mechanically working points, flip-flops and counters which can be fixed on a board and can be connected by simple bricks (straightes, curves, crossings, junctions) to a network (Cohors-Fresenborg 1978).
We have constructed a lesson course (Cohors-Fresenborg/Finke/Schütte 1979) consisting of about 16 lessons, in which pupils learn to build networks as concrete representations of the mathematical idea of periodically counting automata. Those are used in daily life in sorting machines or in selling machines for tickets and stamps. These curriculum elements were tested with several thousand pupils mainly of grade five. It was really astonishing, how successful quiet a lot of these young pupils were in constructing and anylyzing such automata networks.

The registermachine was invented as a microworld to understand imperative programming languages like PASCAL (Cohors-Fresenborg 1987a). It fits to the idea of computing networks constructed with the material dynamic mazes. The registermachine has been used to introduce the concept of function on the basis of algorithms (Cohors-Fresenborg/Griep/Schwank 1982).

Textbook

During our project we have worked out a small textbook for the pupils and a detailed handbook for the teacher (Goldberg 1987). Both are use our experience teaching the concept of function on the basis of algorithms (Cohors-Fresenborg/Griep/Schwank 1982). A deeper insight into the valuue, which play the different forms of representations for the construction of suitable mental models in the deaf pupils' mind, can be got from the design of the two clinical interviews (Cohors-Fresenborg 1987b, appendix page U3-U32).

SCIENTIFIC EXPLANATIONS

In the following we will try to explain the pupils' success in our curriculum project. The research on algorithmic concept formation has worked out three dimensions: the role of different forms of representation (Cohors-Fresenborg 1986) as well as the existance of individual different cognitive structures (Schwank 1986) and cognitive strategies of pupils (Kaune 1985, Marpaung 1986).

The material 'dynamic mazes' plays an important role as a non-verbal communication device in the sense of Lowenthal (1982). The non-verbal aspect of mathematical concept formation is one important reason for the pupils' success. A remarkable contribution to understand the role of language in the process of algorithmic and mathematical concept formation is the work of Lowenthal and Saerens with an aphasic child using dynamic mazes and formal systems (Lowenthal 1985). Our work in curriculum construction was supported by first attempts to explain the pupils' process of concept formation by research in the field of cognitive science.
In the meantime the different forms of representing algorithmic concepts are further analysed (Cohors-Fresenborg 1986). Especially after the investigations of Kaune (1985) these forms are now not regarded to form a hierarchy as it has be seen before. The use of dynamic mazes for concept formation is not only explaned by its non-verbal approach but merely by the fact, that constructing computing networks with the dynamic mazes implements a philosophy of thinking which fits in a very good manner to the cognitive structures of some pupils (Schwank 1986). The dynamic mazes support a thinking in the terms of functioning (functional reasoning), on the contrary a succesful use of dynamic mazes needs a certain ability in this kind of thinking. Only by this it can be explained that there are some pupils which could'nt solve debbuging-problems which were presented in the form of dynamic mazes, but they

were able to solve the analogous problem which was presented in the form of formal computer programs for the registermachine. These individual preferences can be developped in such a manner that pupils by themselves start a procedure to translate the problem from the computing network into a formal program, solve it in this representation and translate it back to the world of computing networks. That means, those pupils are more successful in solving problems using the formal representation of a computer program, which fits better to predicative thinking than the computing networks, allthough they had to organize twice a translation process. For the first time such a phenomenon was reported by Marpaung (1986, page 98).

Mathematical concepts are defined and learned inside a framework of concepts which occur in an axiomatic system of mathematics as free varibals and for which there must be decleard meaning by interpretation. From the psychological point of view there must be given evidence for these fundamental concepts. In the usual way of constructing mathematics these fundamental concepts are of predicative or set theoretical nature; the other mathematical concepts, for example the concept of function, are then introduced by explizit definitions. Concerning the field of teaching and learning concepts the verbal orientated predicative way of concept formation fits to this kind of introducing mathematical concepts. On the opposite the concept of function is fundamental in a constructive foundation of mathematics which from its philosophy fits to the use of computers in mathematics. In this case the basis of evidence is a thinking in the terms of organizing actions and of functioning. Concerning the enumerable mathematics both approaches are equivalent by principle. But this does not mean that they are equal from the psychological point of view. These different possibilitys of foundation of mathematics show, that it is very naive but often found that there exists a unique mathematical kernel of a concept. It may be quite useful inside a given framework of mathematical foundations but it is of no value if one is concerned with a new orientation of the fundaments of mathematics curriculum.

From the cognitive science point of view Schwank (1986) has pointed out that corresponding to these two different mathematical approaches there exist analogous differences in the cognitive structures of human

beeings: the predicative versus the functional structure of thinking. Casestudies with 9 of the deaf pupils have shown, that the destinction between pedicative and functional cognitive structure is useful to explain the observed individual differences in the process of concept formation (Cohors-Fresenborg 1987b, appendix page U39-U42). In a casestudy with the deaf boy Dietmar it could be shown by Schwank (1986), that there exists a deaf, who preferes extremely the predicative cognitive structure, allthough language seems to be the natural tool to use it. This boy therefore was all the more handicapped,because his cognitive structure needs an elaborated language. The teachers regarded him as not so intelligent. And in our action orientated course using dynamic mazes he was not successfull (because ,as we know now, we used them according to functional thinking). But in the casestudy it could be shown, that this boy could really understand the programming language of the registermachine, when this was introduced in a way according to his predicative cognitive structure. In the following it was possible to leed him to a remarkable understanding and success in algebra by a sophisticated use of denominations and formal symbols.
This example shows, that the role of language in mathematical concept formation processes, especially of deaf, must be further analyzed.

REFERENCES

COHORS-FRESENBORG,E.(1978): Learning Problem Solving by Developing Automata-Neworks, in: Proc. 2nd Conf. for PME, COHORS-FRESENBORG/ WACHSMUTH (Eds.), Osnabrück

COHORS-FRESENBORG,E./FINKE,D./SCHUTTE,S.(1979): Dynamic Mazes (9 booklets for programmed lessons), Osnabrücker Schriften zur Mathematik, U 11-19, 11A-19A (in English), U 21 (teacher's handbook, in German), University of Osnabrück

COHORS-FRESENBORG,E.(1986): On the representation of algorithmic concepts. In: LOWENTHAL, F. et alii, Pragmatics and Education. Plenum Press, New York, pp.205-221

COHORS-FRESENBORG,E.(1987a): The benefit of Microworlds in Learning Computer Programming. In: BÖRGER,E. (Ed.), Computation Theory and Logic, Lecture Notes in Computer Science, vol. 270, pp.88-100

COHORS-FRESENBORG,E.(1987b): Algorithmisches Denken im Mathematikunterricht mit Hörgeschädigten, Abschlußbericht über einen BLK-Modellversuch, Universität Osnabrück (unpublished)

COHORS-FRESENBORG,E./GRIEP,M./SCHWANK,I.(1982): Registermachines and Functions - A schoolbook introducing the concept of function on the basis of algorithms. Osnabrücker Schriften zur Mathematik, U 22E, 22LE (in English), U 25 (teacher's handbook, in German)

BEST COPY AVAILABLE

COHORS-FRESENBORG,E./STRUBER,H.J.(1982): The Learning of Algorithmic Concepts by Action - A Study with Deaf Children, in: LOWENTHAL,F. et alii (Eds.): Language and Language Acquisition, New York, pp.95-106

GOLDBERG,A.(1987): Programmieren in der RM-Sprache und in PASCAL, Aufgabensammlung für Klasse 8 (Band 1), Didaktische Analyse und methodische Ausarbeitung einer Unterrichtsreihe (Band 2) Universität Osnabrück (unpublished)

JOHNSON-LAIRD,P.N.(1983): Mental Models, Cambridge

KAUNE,C.(1985): Schüler denken am Computer - Eine Untersuchung über den Einfluß von Repräsentationsformen und kognitiven Strategien beim Konstruieren und Analysieren von Algorithmen, Osnabrück

LOWENTHAL,F.(1982): Example of Auxiliary Formalisms used to help the evelopment of Children's Logical Thinking, in: LOWENTHAL,F. et alii (Eds.): Language and Language Acquisition, New York, pp.113-121

LOWENTHAL,F.(1985): Non-Verbal Communication Devices in Language Acquisition, Revue de Phonétique Appliquée, vol. 73-75, pp.155-166

MARPAUNG (1986): Profile indonesischer Schüler beim Umgang mit Algorithmen und ihre Analyse, Osnabrück

SCHWANK,I.(1986): Cognitive Structures of Algorithmic Thinking, in: Proc. 10th Conf. for PME, Univ. of London Inst. of Education (Ed.)

THE EFFECT OF ORDER-CODING AND SHADING OF GRAPHICAL INSTRUCTIONS ON THE SPEED OF CONSTRUCTION OF A THREE-DIMENSIONAL OBJECT

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ABSTRACT

There are many situations, both in the classroom and in real life, in which a three-dimensional object must be constructed from diagrams.

In the present research, the efficiency of high school students was examined as they constructed a SOMA cube from its components, using instructions consisting of different types of isometric drawings. (The SOMA cube puzzle consists of seven polycubical pieces which fit together to form a cube.) The graphical instructions consisted of an 'exploded view' and a sequenced set of views, each showing a new component being added to the structure. Each type of instruction was presented in a "clear" form, so that all faces, edges and vertices had the same prominence, and in a form in which drawings were "shaded" in such a way as to suggest depth. A cue consisting of a clear exploded drawing, which was coded for order of operation, was provided also.

The sample consisted of 48 Year 7, 42 Year 9 and 27 Year 11 boys, subjects being examined one at a time. The proportion of Year 7 subjects who were able to complete the task successfully was so small that no comparisons among the mean times to successful completion were made for this age-group. For the other years, the results suggest that sequenced drawings are more effective than exploded views that are not coded for order. For Year 9, all order-coded illustrations taken together (whether exploded views or not) were found to be more effective than both illustrations that were not coded for order. Again, for Year 9 but not for Year 11 subjects, the clear exploded view that was not coded for order was not as efficient as the clear, order-coded exploded view. For neither age-group was any difference in effectiveness found between clear illustrations and shaded illustrations.

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THE EFFECT OF ORDER-CODING AND SHADING OF GRAPHICAL INSTRUCTIONS ON THE SPEED OF CONSTRUCTION OF A THREE-DIMENSIONAL OBJECT

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BACKGROUND

There are many situations, both in the classroom and in real life, in which a three-dimensional object must be constructed from diagrams. Such diagrams may be in the form of plans or elevations ("orthogonal views" or "sections"), they may be perspective or isometric drawings, or they may take the form of "exploded views" of the sort found in automobile handbooks.

In the construction industry, plans and elevations are common formats for transmission of information about the object being manufactured, and in Geography, maps which represent elevation as well as layout are used extensively. In Mathematics classes dealing with topics such as mensuration of three-dimensional objects and solid geometry, however, isometric or perspective drawings tend to be used most often (Goddijn and Kindt, 1985). Instructions which accompany household "assemble-it-yourself" articles, also, frequently use isometric or perspective drawings as illustrations; sometimes such drawings are coded so that the user knows the order in which the parts are assembled.

Recent research (Cooper and Sweller, in press) has shown that Grade 7, Grade 9 and Grade 11 students find it much easier to assemble cubes into simple polycubical structures when following instructions which include prototypes or isometric drawings than from instructions which use plans or elevations. Furthermore, they found no difference in efficiency between the use of isometric drawings and prototypes of the structures to be assembled. This finding is in accord with Metzler and Shepard (1974) who suggested that isometric drawings are internalized in the same way as the structure itself.

In the present research, we examined the speed with which high school students could construct a solid from its polycubical components, when using different types of instructions based on isometric drawings. When employed with instructions for the assembly of an object, such drawings may carry an indication of the order in which each component is put into position. Although such "order coding" provides additional information, and may therefore increase cognitive load, it seemed probable that the format would be more efficient because such additional information allows better management of the task and thus reduces the need to employ trial-and-error methods. Exploded views are, of course, isometric in nature. It was conjectured that instructions using order-coded exploded views would be more efficient than those in which the diagrams were not coded in this way.

Another type of instruction, in which both isometric drawings and order coding are combined, takes the format of an ordered series of views, each showing a new component being added to the structure (see Figure 4). It was conjectured that this type of instruction would be more efficient than that using the exploded-view type of isometric drawing.

Isometric drawings may be "clear" in the sense that all faces, edges and vertices have the same prominence (see, for example, Figure 4); on the other hand, they may be "shaded" in such a way as to suggest depth (as may be seen in Figure 5). Shading of this sort was used by Gaulin (1985) and by Izard (1987). As there appears to be no research evidence that either shaded or unshaded drawings are the easier to interpret, one may ask why such a device is employed at all.

METHOD

Materials

The materials consisted of the seven components of a standard SOMA cube and five cards, each bearing an illustration. (The SOMA cube puzzle consists of seven polycubical pieces which fit together to form a cube.) Each illustration was designed to accompany the following instructions:

> These wooden pieces fit together to form a cube. Can you put this puzzle together? The drawings on this card are provided to help you.

The respective cards and the format of their illustrations were as follows:

Card 1:	Clear, uncoded exploded view (see Figure 1)
Card 2:	Shaded, uncoded exploded view (see Figure 2)
Card 3:	Clear, order-coded exploded view (see Figure 3)
Card 4:	Ordered sequence of cumulative isometric drawings; clear format (see Figure 4)
Card 5:	Ordered sequence of cumulative isometric drawings; shaded format (see Figure 5)

Procedure

Samples were drawn from grade 7, grade 9 and grade 11 classes in a large high school. Subjects were examined one at a time. Each subject was seated at a table and presented with the seven SOMA cube components and an illustrated instructions card which was randomly selected from the five available.

The researcher recorded the time taken from the instant the subject viewed the material to the instant when the task was completed. A time of 10.00 minutes was recorded for subjects who had not completed the task by that time.

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Figure 1 Clear, Uncoded Exploded View

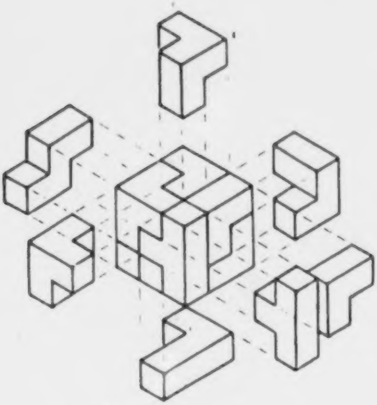


Figure 2 Shaded, Uncoded Exploded View

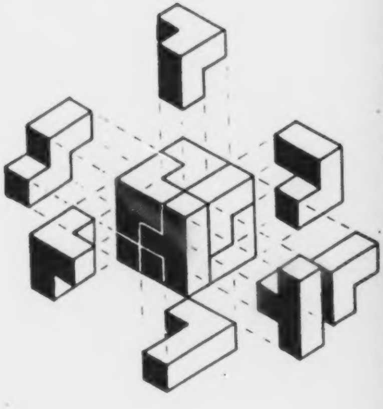


Figure 3 Clear, Order-Coded Exploded View

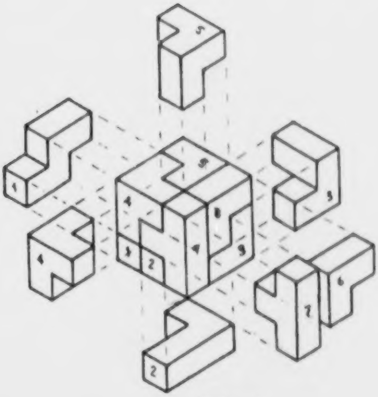

5
1
9
4
4
8
3
3
2
4
3
6
7
2

Figure 4 Ordered Sequence of Cumulative Isometric Drawings, Clear Format

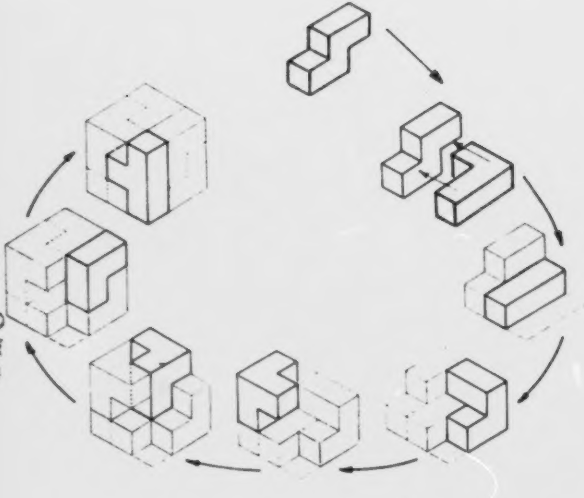
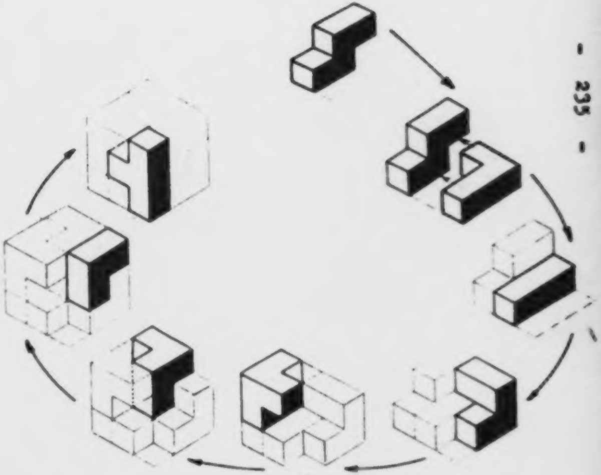


Figure 5 Ordered Sequence of Cumulative Isometric Drawings, Shaded Format



RESULTS

For each grade-type combination, the number and percentage of subjects who were able to complete the task correctly in a time not exceeding ten minutes is shown in Table 1.

Table 1 Numbers and percentages able correctly to complete the task

card	type of illustration	grade 7 N	grade 7 n	grade 7 %	grade 9 N	grade 9 n	grade 9 %	grade 11 N	grade 11 n	grade 11 %
1	exploded view (clear)	8	6	75.0	9	7	77.8	7	7	100.0
2	exploded view (shaded)	7	3	42.9	9	7	77.8	6	4	66.7
3	order-coded exploded view	8	2	25.0	9	8	88.9	6	6	100.0
4	sequenced drawings (clear)	9	5	55.5	8	7	87.5	6	4	66.7
5	sequenced drawings (shaded)	8	3	37.5	9	9	100.0	6	6	100.0

N: number attempting task
n: number succeeding at task

Because the proportion of Grade 7 subjects who were able to complete the task successfully was so small, it was decided not to make any comparisons among the mean times to successful completion for this sample.

The number of subjects in Grade 9 and Grade 11 successfully interpreting each type of illustration and the mean and estimated standard deviation of their times to completion of the task are given in Table 2.

Table 2 Means and estimated standard deviations of times to successful completion

card	type of illustration	Grade 9 n	Grade 9 mean	Grade 9 sd	Grade 11 n	Grade 11 mean	Grade 11 sd
1	exploded view (clear)	9	5.71	2.53	7	2.88	0.96
2	exploded view (shaded)	9	4.75	2.22	4	4.16	1.78
3	order-coded exploded view	8	3.20	2.38	6	3.42	1.79
4	sequenced drawings (clear)	7	2.80	1.83	4	2.11	1.42
5	sequenced drawings (shaded)	9	3.74	2.58	6	1.94	1.07

n: number succeeding at task
sd: estimated standard deviation

It was expected that

1. sequenced drawings would be more effective than exploded views;
2. illustrations indicating order of operations would be more effective than those not showing order;
3. 'shaded' illustrations would be more effective than clear illustrations;
4. the order-coded clear exploded view would be more effective than the clear exploded view bearing no indication of order of operations.

Contrasts corresponding to the above planned comparisons were therefore defined as follows:

contrast 1:- average of means for Cards 1 and 2 (non-order-coded exploded views) **minus** average of means for Cards 4 and 5 (sequenced drawings)

contrast 2: average of means for Cards 1 and 2 (non-order-coded exploded views) **minus** average of means for Cards 3, 4 and 5 (order-coded exploded view and sequenced drawings)

contrast 3: average of means for Cards 1 and 4 (clear exploded view and clear sequenced drawings) **minus** average of means for Cards 2 and 5 (shaded exploded view and shaded sequenced drawings)

contrast 4: mean for Card 1 (non-order-coded clear exploded view) **minus** mean for Card 3 (order-coded exploded view)

The contrast estimates and their estimated standard errors are presented in Table 3.

Table 3 Contrast estimates and their estimated standard errors

	Grade 9		Grade 11	
contrast	contrast estimate	estimated std. error	contrast estimate	estimated std. error
1	1.96*	0.81	1.49*	0.63
2	1.98*	0.73	1.03	0.56
3	0.01	0.81	-0.55	0.63
4	2.51*	1.14	-0.54	0.78

*p<0.05

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The estimates of contrasts 1, 2 and 4 are significantly different from zero at the 0.05 level for Grade 9; for Grade 11, only contrast 1 shows significance.

The results of the above statistical analysis suggest that instructions using sequenced drawings of the type illustrated in Figures 4 and 5 are more efficient than those using exploded views that are not coded for order. For Grade 9, all order-coded illustrations taken together (whether exploded views or not) were found to be more effective than both illustrations that were not coded for order. Again, for Grade 9 but not for Grade 11 subjects, the clear exploded view that was not coded for order was not as efficient as the clear, order-coded exploded view.

For neither grade was any difference in effectiveness found between clear illustrations and shaded illustrations. This result may be explained in terms of Metzler and Shepard's statement that isometric drawings of three-dimensional polycubical structures are internalized as if the structure itself were being viewed - if a clear isometric drawing is as effective as the structure, then a shaded isometric drawing can hardly be more effective. This finding may not hold, of course, for illustrations of more complex structures.

REFERENCES

Cooper, M. and Sweller, J. (in press), Secondary school students' representation of solids, **Journal for Research in Mathematics Education**

Gaulin, C. (1985), The need for emphasizing various graphical representations of 3-dimensional shapes and relations, **Proceeding of the Ninth International Conference for the Psychology of Mathematics Education (vol. II)**, Noordwijkerhout, 53-71

Goddijn, A. and Kindt, M. (1985), Space geometry doesn't fit the book, **Proceeding of the Ninth International Conference for the Psychology of Mathematics Education (vol. I)**, Noordwijkerhout, 171-182

Izard, J. (1987), **Acquisition of spatial skills under vrious conditions of instruction**, paper presented at the First Joint Conference of the Australian Association for Research in Education and the New Zealand Association for Research in Education, Christchurch, 1987

Metzler, J. and Shepard, R.N. (1974): Transformational studies of the internal representation of three-dimensional objects; in R. Solso, **Theories in Cognitive Psychology; The Loyola Symposium**, Potomac, Erlbaum, 147-201

NEW CONTEXTS FOR LEARNING IN MATHEMATICS.

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This paper presents a work in progress report of an exploratory study investigating the initial responses of 25 four year old children to a modified version of LOGO and the relationships between learning behaviour and the social context in which learning occurs. The research forms part of a wider study, involving 450 families, of the social context of mathematics learning in early childhood. A description is given of the patterns of response by the children and gender differences in interaction with computational medium are discussed. The results of the study indicate that the effects of socio-cultural factors are heightened in informal learning situations where self directed activity is encouraged. Also, that computational media are a potentially powerful educational tool when used to structure incidental learning.

In developed countries, increased computer use has resulted in both qualitative and quantitative changes in the information available to all sections of the population. Computers and computational media provide new objective and social contexts in which young children may acquire mathematical knowledge and use it in purposeful activity. There is a need for careful investigation of the changing contexts in which learning in mathematics occurs.

An interactive approach to cognitive activity.

There is increasing acceptance among cognitive theorists that children construct their own knowledge through reflective activity associated with their experience of the objective and social environments. Constructivist theorists (for example, Cobb (1986) and Duckworth (in press)) have recently described this process in some detail. Cobb (1986) describes the interaction between perceptual information and prior knowledge as an individual observes mathematical aspects of reality as follows:

> Mathematical structures are not apprehended, perceived, intuited or taken in, but are instead reflectively abstracted

> from sensory motor and conceptual activity. Consequently the adult observer who "sees" arithmetical knowledge "out there" is consciously reflecting on structures that he or she has imposed on reality.

Cobb and Duckworth, like Piaget (1953), view cognitive development as a generative process in which a child adaptively responds to experience of the objective environment.

Some soviet psychologists ((Luria (1973), Vygotsky(1978), Leont'ev(1981)) and Olsen (1987) have placed rather more emphasis on the relationships between cognitive and social acts, particularly in relation to the development of abstract concepts such those in science and mathematics. Luria (1973) and Leont'ev (1981), in particular, have explored the ways in which cognitive activity is constrained by the social context as well as physical aspects of the environment. Crawford (1986), in a study of children's mathematical behaviour, found some evidence supporting their position. It appears that differing cognitive functions are used to process information according to how an individual perceives his/her needs and interprets the goal of a task. For example, if a task is perceived as involving self directed, creative activity, then metacognitive processes are likely to be used to direct conscious reflection and critical evaluation of efforts to "make sense of" a situation. In contrast, if the social context is interpreted as requiring the following of directions, an individual concentrates on the use of cognitive processes associated with accurate imitation. The latter processes are not normally available for conscious reflection. Luria (1982) states that the use of different functions results in qualitatively different epistemological outcomes. Conscious reflection is associated with concept development. Careful and repeated imitation results in well automatised routines that are not easily accessable for conscious review.

The subjects in Crawford's (1986) study were flexible about adapting their behaviour when the social context for mathematical problem solving was explicitly redefined. However, in many elementary school classrooms in Australia there is an implicit (and often explicit) requirement to follow the directions of the teacher. Such a social context mitigates against purposeful cognitive activity in mathematics.

New contexts for learning.

Booth (1984) has described how the spontaneous pattern making of young children with paint and paper is influenced by the physical constraints (e.g. paper shape) of the activity. Technological environments now make possible new contexts for learning and also imply the need for new kinds of knowledge. Routine procedures are increasingly a function of machines. There is less demand for humans to accurately perform highly automatised skills. Higher order thinking - planning, trouble shooting, and creative problem solving - are necessary for future purposeful use of technology. Computational media also provide new kinds of information. For example the dynamic visual representations of abstract mathematical ideas available in ELASTIC (developed by A.Rubin BBN laboratories) have not been available to students of the past. Computers also provide new social contexts for mathematical enquiry. Now a machine rather than an adult sometimes serves as a source of information. Also, ideas represented on a screen, as occurs when children use LOGO, are available for public scrutiny and discussion. Finally, and most important, computational media are highly structured and provide new constraints as the basis of incidental learning during use.

Learning and computers at pre-school.

As part of a larger study of the social context of mathematical learning in young children in 450 families, 25 pre-school children were observed during their initial contact with a modified LOGO programme (Little Logo). At the beginning of the programme the children's ages ranged from 54 to 60 months.

The time spent at the computer was described to the children as "drawing on TV" in an effort to encourage association with other activities in which free expression and creativity were encouraged. The children were explicitly encouraged to explore the new medium freely. In keeping with the rest of the informally organised pre-school programme use of the computer was not mandatory but during the twelve weeks all children were encouraged to use the computer either singly or in small groups. All but one child used the computer on a weekly basis. An adult in attendance kept running records of the interaction for the first half of the twelve week programme and assisted when technical difficulties were encountered. Technical information about the programme was presented in three stages. First children were introduced to the commands of the modified LOGO programme (R L F C then B D U T H ?). After four weeks experience of using the commands and observing the effects produced on the screen, children were encouraged to use the same commands to direct a turtle robot on the floor. After two sessions with the robot the children reverted to "drawing on TV".

Records were kept of the time spent by each child, knowledge of commands (use of left and right were noted separately), evidence of planning, whether the child initiated or followed ideas. Printed records of the children's "drawings" were also kept. Under such informal conditions the following consistent developmental sequence emerged:

1. Initial random and impulsive experimentation with the commands. In the new medium this stage appears similar to the scribble stage described by Booth (1984).
2. Initial attention in purposeful investigation of the new medium centred around the two features most obviously different from the topogical constraints of paper and pencil. These were: attemps to predict where the "turtle" was after using the H (hide) command and also experimentation with continuous use of F (forward) until the "turtle" reappeared on the screen.
3. A gradually increased focus on the horizontal and vertical axes. Horizontal and vertical lines usually represented the children's first attempts to control direction. The terms quickly became part of the vocabulary used to discuss the effects obtained on the screen.
4. Isolated geometric shapes. Usually rectangles, squares, triangles and "circles" (closed curves). This stage generated considerable discussion about the repetitive use of the F (forward), L (left) and R (right) commands. Methods of producing shapes of different sizes were exchanged and discussed.

5.Planned filling of the available space. For boys this stage often involved use of geometric patterns in a manner similar to that described by Booth (1984), for girls there was less interest in repetition and more interest in detailed representation of people and events.

The sessions in which the children used the turtle robot were of interest. All children were initially startled and delighted when they discovered that the commands they had learned could be used to move the turtle on the floor. The new experience was clearly associated (particularly by the girls) with experience playing with soft toys and dolls. A "house" and "shops" were quickly constructed with cardboard boxes and the robot was moved about. There was a marked increase in imaginitive play and planning with work on the video monitor after children had had experience in controlling the turtle robot on the floor. The robot was slow and cumbersome and most soon got bored with it. However the experience was very effective as a means of shifting the children's use of the medium from experimentation to more planned and extended activity.

There were several differences in the ways that boys and girls approached the use of the computer. Sperber & Wilson (1986) would describe the differences as differences in the "cognitive environments" that boys and girls brought to the task. These can be summarised as follows:

1. Greater initial enthusiasm for computer use by boys.
2. Boys were eager to explore the limits of the new medium by themselves whereas girls initially requested assistance more often. (girls showed greatly increased planning behaviour and initiative when left without supervision.)
3. Once competence in controlling the medium had been attained, there were qualitative differences in the kinds of representation by boys and girls. In general girls attempted to represent people and activities in as much detail as the medium allowed, boys were more interested in exploration of and rearrangement of the shapes they had created.

By the end of the programme there were no overall differences between boys and girls in enthusiasm for computer use or in knowledge of commands. Most children were able to competently discuss their activity in terms of distance, direction, horizontal/vertical, left/right, shape and size. However, the qualitative differences in the use of the medium resulted in different expressions of frustration with the limitations of the restricted LOGO programme. Boys were most interested in the possibilities for translation (and in some cases rotation) of shapes and repetition of procedures. In contrast, the girls were able to clearly articulate their frustration with limitations imposed by the L and R commands (set at 30 degrees) and their need for more precision in forming angles. The results support Leont'ev's notion that cognitive activity is defined by the needs and goals associated with a subjective interpretation of a task. The gender differences described above were consistent with significant differences in parental expectations and activity choice found in the wider study.

An adult was in attendance for all sessions of the computer activity for the first half of the programme. Although explicit instructions were given that children should freely explore the medium with a minimum of adult direction, it was clearly difficult to resist the temptation to "help". Many children, girls in particular, actively sought help and felt most comfortable in dependent learning situations. The situation remained unresolved until illness resulted in the children managing the computer activities by themselves for a week. There was an increase in active discussion about the activities and in the planning and complexity of the resultant "drawings". From that time, adult assistance in the programme was reduced to one session per week.

Conclusions.

The results of this small exploratory study should not be generalised. However, the study has highlighted the significant influence of socio-cultural experience on learning in informal settings and the way in which computational media provide new and influential contexts for incidental learning in young children. It is clear that purposeful creative activity with computational media incidentally direct children's attention to mathematical aspects of the environment. However, the socio-cultural associations and perceptions of "appropriate" behaviour strongly influence the course of self directed learning. Even at four years of age, gender differences in the "cognitive environments" that children brought to the activity were evident. Boys and girls increasingly used the medium to achieve different goals and to explore different interests. As a result, they paid attention to and discussed reflectively different aspects of the medium.

The study raises a number of questions for future research. Some of these are:

> Are the socio-cultural effects associated with achievement in mathematics exacerbated or ameliorated by less formal settings and self directed mathematical enquiry?
>
> Can computational media be used to enhance mathematical understanding while individuals persue their different goals and interests?
>
> How do the structures of a computational media influence incidental learning during use by students?

With the increased use of micro-computers in schools and new developments in educational software changes in both the social and physical contexts of learning in mathematics seem inevitable. However if these chnages are to have positive educational outcomes, there is an urgent need for research directed at increasing our understanding of the above questions.

References.

Booth, D. " Aspects of logico-mathematical intuition in the development of youg children's spontaneous pattern painting." in the Proceedings for the Eighth International Conference for the Psychology of Mathematics Education, Sydney, Australia, August 1984, 224-232.

Cobb, P. "Making Mathematics: Children's Learning and the Constructivist Tradition." Harvard Educational Review 56(3),1986, p302.

Crawford, K.P. "Simultaneous and successive processing, executive control and social experience: Individual differences in educational achievement and problem solving in mathematics." An unpublished thesis submitted in fulfilment of the degree of Doctor of Philosophy, University of New England, 1986.

Duckworth, E. Opening the World. Lawrence Erlbaum & Associates, in press.

Leont'ev, A.N. "The Problem of activity in psychology" in J Wertsch (ed.) The Concept of Activity in Soviet Psychology, U.S., M.E. Sharpe, 1981.

Luria, A. R. The working brain. Penguin, 1973.

Luria, A.R., Language and Cognition (ed.) J.V. Wertsch, U.S., Wiley, 1982.

Olsen S. The politics of mathematics education. Kluwer Academic, 1987

Piaget, J. The Origin of intelligence in the child. 2nd Edition Routledge Kegan Paul, 1953.

Sperber, D & Wilson, D. Relevance: Culture and Cognition. U.K. Basil Blackwell, 1986.

Vygotsky, L.S., Mind in Society, M. Cole et al. (Eds), U.S. Harvard University Press, 1978.

QUELQUES DEVELOPPEMENTS RECENTS DES RECHERCHES SUR LA DISCUSSION AUTOUR DE PROBLEMES

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DISCUSSION ABOUT PROBLEMS : RECENT TRENDS IN RESEARCH AND ANALYSIS

This study refers to the conception, the realization, the observation and the analysis of a teaching approach for mathematics, we called "Discussion About Problems" (D.A.P.), and which was experienced for two years (from '85 to '87). It is based on the direction (by the teacher) of a discussion among students (adults) about the statements they make regarding a problem they must have yet prepared.

The ongoing research focuses on analysis of video-registered observations, and aims at determining characteristics of D.A.P. and discovering some rules of action and interpretation the teacher uses when conducting a discussion.

One of the results we obtained is that a procedural description based on "parenthesis" (sub-discussions) is irrelevant. Hence, we are working on a new "theatrical" point of view, which is based on mathematical contents and communicational side together.

0. INTRODUCTION

Comment penser une démarche didactique qui puisse favoriser l'aquisition d'un certain nombre de connaissances en mathématiques (de divers niveaux), par l'intériorisation (individuelle) d'un débat (collectif)? La conception théorique, la mise en place, l'observation et l'analyse d'une telle démarche (que nous avons appelé "Discussion Autour de Problèmes" : D.A.P.) sont l'objet d'une recherche menée dans le cadre des séances de travaux dirigés d'un enseignement de remise à niveau pour adultes. A titre indicatif, le programme débute avec les calculs sur les puissances et les fonctions affines, et se termine par les coniques, l'intégration et les nombres complexes (1).

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(1) Le lecteur pourra se faire une idée de ce que peut être une séance de D.A.P. en se référant à la description schématique qui figure en annexe (N°1).

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Un des soucis fondamentaux de notre recherche sur la D.A.P. consiste à éviter, autant que possible, que nos constructions théoriques (a priori) n'altèrent notre interprétation (a posteriori) des observations. Toutefois, on ne peut faire un tel travail d'interprétation comme si l'on était vierge de tout a priori. Il importe donc d'élucider au mieux notre théorisation initiale et son influence sur la conduite de l'expérimentation et sur l'interprétation des observations recueillies. Pour éviter de "prendre nos désirs pour des réalités", il faut mettre en rapport ce que nous voulions (initialement) qu'il se produise, et ce que nous pensons (maintenant) qu'il est arrivé.

Par ailleurs, la scientificité d'une expérimentation didactique nous semble dépendre, entre autres, du statut accordé à l'analyse des décalages entre "intentions" (telles qu'on peut les reconstituer) et "réalisations" (connues par les observations et les analyses). Considérer ces décalages comme une "dérive" inévitable, qu'il s'agirait de réduire par approximations successives, nous parait risquer de tirer l'étude vers le domaine de l'innovation pédagogique. Ce n'est pas un mal en soi, mais cela correspond à un affaiblissement de la vigilance épistémologique (1) et sort du domaine de la recherche didactique. Au contraire, nous considérons l'interprétation de ces décalages comme un moyen privilégié d'en connaître autant sur nos intentions que sur les contraintes externes (institutionnelles) et internes (didactiques) de la situation mise en jeu.

Encore faut-il disposer des moyens de prendre connaissance, de la manière la plus objective possible, du déroulement effectif des séances ; ou, si l'on préfère, pouvoir passer du commentaire à l'analyse, c'est à dire à la compréhension de la "logique" des discussions.

La recherche d'une telle méthode d'analyse a commencé dès la deuxième année de l'expérience (1986-1987) et n'est pas terminée. Toutefois à l'occasion de la phase initiale de ce travail d'analyse, consistant à "découper" les décryptages, les problèmes rencontrés et les solutions envisagées nous indiquent quelques résultats concernant les interprétations des situations de type D.A.P.

1. DESCRIPTION PROCEDURALE

Nous avions voulu tester une méthode d'analyse des séances de D.A.P. reposant sur le principe "procédural" suivant :

> Une D.A.P. peut être considérée comme une procédure complexe de résolution d'un problème, procédure dont le déroulement nécessite d'accomplir des tâches dont certaines sont elle-mêmes des procédures complexes de résolution d'un sous-problème, et ainsi de suite.

Ce principe pourrait se schématiser comme suit :

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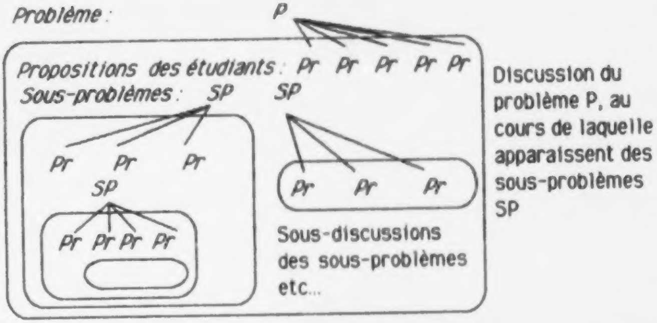
(1) En prenant ce terme dans le sens où l'emploient Bourdieu et al. (1983)


Problème :
P
Propositions des étudiants : Pr Pr Pr Pr Pr
Sous-problèmes : SP SP
Pr Pr Pr
SP
Pr Pr Pr Pr
Pr Pr Pr
Sous-discussions des sous-problèmes etc...
Discussion du problème P, au cours de laquelle apparaissent des sous-problèmes SP

C'est cet emboîtement de sous-discussions dans une discussion, qui devait constituer un principe descriptif permettant de représenter schématiquement (comme ci-dessus) les D.A.P. sous forme arborescente. En d'autres termes, dans cette description, pour gérer la discussion (ce qui est la tâche de l'enseignant) il fallait contrôler la profondeur des "parenthèses" (c.a.d. des sous-discussions de résolution de sous-problèmes). Ce terme de «parenthèse» avait pour origine, entre autres, l'interprétation spontanée des difficultés réellement ressenties pendant la gestion pratique des séances.

Lorsqu'une discussion devenait "illisible" pour le maître et pour les élèves, à cause d'un trop grand nombre de questions imbriquées les unes dans les autres, il était tentant d'interpréter cela comme l'"ouverture" (par l'enseignant) d'un trop grand nombre de "parenthèses".

Un tel schéma reste à peu près vide de sens tant qu'on n'en a pas élucidé les mots-clefs ("discussion", "problème", etc...) La place manquant toutefois ici pour le faire, nous ne pouvons que renvoyer le lecteur à DROUHARD et PAQUELIER (1987) (ainsi qu'aux publications du Groupe de Recherche sur la Didactique de l'Enseignement Supérieur -cf. bibliographie- pour une présentation des situations de "Débat Scientifique", qui présentent des points communs avec celles de D.A.P.). Toutefois, figure en annexe (A2) un rappel de quelques-unes des positions théoriques qui sont à l'origine de notre démarche.

Or ce schéma descriptif, qui reflète assez fidèlement un stade antérieur de nos conceptions de la D.A.P., s'est révélé mal adapté à la description du déroulement des séances.

1) Lorsqu'on essaye d'analyser un décryptage selon ce schéma, on se trouve devant un certain nombre de discussions ne correspondant à aucun sous-problème explicite. Pour sauver la description, on est amené à supposer l'existence de sous-problèmes hypothétiques.

2) Plus fondamentalement, cette première conception de la D.A.P. était caractérisée par une relative "étanchéité" des niveaux (mathématique, métamathématique, méthodologique, etc...) Elle supposait, en effet, plusieurs types de D.A.P. : discussion sur les solutions mathématiques d'un problème mathématique, discussion sur des méthodes proposées en réponse à un problème méthodologique, etc...

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De fait, les problèmes soumis au étudiants étaient censés ne comporter qu'un ordre de difficulté à la fois (1). Cette première conception ne prenait donc pas en compte la possibilité d'interventions à plusieurs niveaux au cours d'une même discussion.

Or, une remarque de méthode, énoncée à un moment donné d'une discussion portant sur un problème mathématique, peut d'un coup rendre sans objet toutes les propositions de solutions qui l'ont précédée, et donner un cours nouveau au débat, en posant des contraintes sur toutes celles qui suivront. Cette possibilité qu'un seul énoncé "influence" tous les autres, à tous les niveaux, fait voler en éclats la hiérarchisation de la D.A.P. en problèmes et sous-problèmes, basée sur une analyse "procédurale", qui voulait qu'une fois résolu le sous-problème SP, on "remonte", muni de la solution de SP (qui ferait alors partie du savoir du groupe), au problème P, lequel serait resté "en plan", sans modification, dans l'attente de cette solution de SP .

Cette inadaptation de la description aux observations éclaire du coup le décalage entre les préparations des séances (analysées a priori suivant un tel schéma descriptif, alors implicite) et leur déroulement. Nous avions observé ce décalage, mais en le mettant au compte de la mise en oeuvre inadéquate d'une analyse correcte.

Par ailleurs, nous avons longtemps continué à concevoir l'analyse des séances de D.A.P. suivant le schéma initial, alors que celui-ci avait cessé de correspondre aux observations, et même à l'évolution de la théorisation. Ce fait nous paraît une excellente illustration de la nécessité de porter son attention sur la réalisation effective des dispositifs didactiques.

3. VERS UNE ANALYSE "DRAMATURGIQUE"

Dire que toute intervention peut influencer toutes celles qui suivent, revient à signifier qu'on ne peut réduire la clôture d'une discussion à une "fermeture de parenthèse", dont le contenu entier pourrait être oublié au profit de la seule connaissance visée. Contrairement (peut-être) au "Débat Scientifique", la D.A.P. ne nous paraît pas être essentiellement un simple moyen d'«aborder les problèmes réellement posés sur les concepts enseignés» (G.R.D.E.S., 1987, communication à PME XI). La discussion n'a pas pour fonction d'être oubliée sitôt la connaissance aquise, mais au contraire nous paraît constitutive du sens que les étudiants pourront donner (y compris ultérieurement) à cette connaissance.

Autrement dit, nous sommes amenés à penser que les énoncés (de divers niveaux) apparus lors d'une séance, ont une "histoire", qui est un élément essentiel de leur sens . C'est ici que nous retrouvons la problématique générale de l'"intériorisation du débat". Pour nous, ce n'est pas le seul savoir établi lors du débat qui est intériorisé, mais l'unité formée du savoir et de son histoire.

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(1) Par exemple, certains énoncés portaient sur des objets mathématiques parfaitement connus (notion de multiple, ou équation du premier degré), mais posaient des problèmes de logique ; pour d'autres énoncés, c'était l'inverse.

Toute réflexion didactique sur l'institutionnalisation des savoirs en situation de débat, doit à notre avis tenir compte de cet aspect des choses.

Ceci dit, tout ce qui précède est présenté en "forçant le trait", aussi bien en ce qui concerne la schématisation des situations de D.A.P., que leur opposition avec celles de "Débat Scientifique" (1). En particulier, ce qui vient d'être dit sur l'intériorisation des connaissances doit être modulé en fonction de leur niveau: mathématique, métamathématique, méthodologique, etc...

Maintenant, comment analyser de tels énoncés "à histoire" et rendre compte de la logique des séances ?

Les difficultés d'une interprétation procédurale d'une séance de D.A.P., liées au constat qu'il n'est pas possible d'analyser séparément le contenu d'une séance et son fonctionnement communicationnel, nous ont amenés, entre autres, à explorer la voie d'une analyse que nous qualifions de "dramaturgique". Nous cherchons ainsi à justifier le découpage des séances en "actes", "scènes", "moments" et "transitions", en repérant les changements selon deux niveaux, qualifiés de "problématique" et "dramaturgique" (qui correspondent, dans un certain sens, au plan du "contenu" et à celui des phénomènes de communication).

4. PERSPECTIVES

Dans la mesure où un tel travail consiste à "objectiver" au maximum l'analyse des observations, on prendra garde à ne pas substituer au sens opératoire des mots employés (2) (sens que nous sommes précisément en train d'établir), une signification métaphorique dérivée sans contrôle. Ainsi, le déroulement d'une D.A.P. s'apparente plutôt, par sa forme, à une improvisation collective sur un sujet donné, qu'à la représentation d'une pièce déjà écrite.

Cette remarque vaut également pour une autre piste de recherche que nous explorons à l'heure actuelle, consistant à penser la gestion d'une D.A.P. avec certains concepts tirés des recherches en Intelligence Artificielle (3). De ce point de vue, décrire la gestion d'une D.A.P., c'est mettre en évidence quelques «règles» (d'interprétation et d'action) de l'enseignant considéré comme un «expert» (et ensuite, analyser en quoi une telle description se sera écartée des observations !)

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(1) La démarche de D.A.P. concerne les Travaux Dirigés, tandis que le "Débat Scientifique" porte essentiellement sur les cours : ce qui est valable pour l'un ne l'est pas forcément pour l'autre. L'étude de ces différences fait d'ailleurs partie de notre recherche.

(2) "Actes","scènes","dramaturgie"etc...

(3) En particulier, nous pensons que les descriptions de D.A.P. gagneraient à être envisagées sous forme "déclarative" et non pas "procédurale" (ce qui est à rapprocher de la différence de sens entre les mots «gérer» et «organiser» se rapportant à une discussion).

Enfin, l'analyse des D.A.P. pourra servir à déterminer un "profil" différentiel des séances, ce qui permettra d'expliciter ce qui est caractéristique de cette démarche d'enseignement, par rapport à tout autre type de situation où "les élèves participent".

En résumé, cette recherche pourra permettre d'expliciter les liens entre les choix théoriques effectués, les dispositifs d'enseignement retenus et les pratiques effectivement adoptées, tels que nous pouvons les connaître par l'interprétation des données d'observation.

ANNEXES :

(A1) DESCRIPTION SCHEMATIQUE D'UNE SEANCE DE D.A.P.

Cette description schématique est tirée du décryptage d'une séance qui a eu lieu le 14/02/1987.

1) L'enseignant rappelle le problème, et il demande aux élèves leurs propositions et remarques sur ce sujet (seuls les élèves y ayant effectivement réfléchi avant la séance proposeront des énoncés ; par contre, d'autres élèves participeront aussi à la discussion).

2) Sous la dictée des élèves, le maître écrit au tableau les diverses propositions, éventuellement accompagnées d'un début de justification. Il est à noter que ces interventions se situent à des niveaux différents :

- solution au problème,
- remarques méthodologiques sur la manière de le résoudre,
- questions complémentaires,

etc ...

L'enseignant intervient pour reformuler (ou faire reformuler) les propositions, signaler les rapports entre elles (équivalence, contradiction, etc...), mais ne laisse pas transparaître son avis sur leur validité.

3) Le maître sollicite les interventions des élèves pour attaquer ou défendre les propositions. De même que ces dernières, les nouvelles interventions se situent à des niveaux divers :

- autoréfutation d'une proposition par son auteur,
- énoncé d'une nouvelle conjecture,
- contre-exemple à un autre proposition,
- contestation de ce contre-exemple sur un point de calcul,
- questions de technique mathématique posées à la cantonnade,
- proposition d'une méthode pour résoudre une difficulté,

etc...

Durant cette phase, le maître donne la parole, transcrit au tableau, reformule ou fait reformuler certains arguments. Il met aux voix certaines propositions afin d'amener le plus grand nombre d'élèves à prendre parti personnellement. Il lui arrive d'intervenir directement, en tant que participant au débat - et non pas en tant que maître dont la prise de parole serait le signe d'une institutionnalisation. À ce propos, vouloir que le maître soit réellement "neutre" au niveau du contenu est une exigence sans doute irréalisable, et peut-être même dénuée de sens ou non pertinente. En tous cas, la "neutralité" du maître est une composante variable de la situation : le vrai travail didactique nous paraît consister à comprendre comment le maître "triche", consciemment ou non, avec son statut d'animateur "neutre" du débat.

A d'autres moments, il récapitule, et commente un point de mathématiques ou le cours que prend la discussion. Enfin, il peut proposer lui-même un nouvel énoncé à la collectivité.

4) La (sous-) discussion d'une proposition se clôt généralement lorsque personne ne conteste plus une récapitulation proposée par le maître.

(A2) A L'ORIGINE DE LA DEMARCHE

(d'après le Cahier de Didactique des Maths 49)

« Moins qu'un comportement acquis et consolidé par des années d'études, nous pensions que c'était un certain état d'esprit qu'il nous fallait modifier. Cela supposait d'agir sur l'ensemble du système relationnel (maître - élève - savoir) propre à toute situation didactique. C'est ce que nous avons défini par le triple déplacement d'attitude suivant :

1 Faire passer la communication maître-élève (et aussi inter-élèves) du registre rhétorique (au sens péjoratif) BON/MAUVAIS au registre rationnel VRAI / FAUX.

L'objectif de l'étudiant, lorsqu'il communique une solution à l'enseignant, est davantage de satisfaire ce dernier dont il attend un jugement de valeur (la note) que de le convaincre (cf. ALIBERT et al. 1986, G.R.D.E.S 1987). Ceci s'exprime, notamment, dans une attitude "juridique" (LACOMBE, 1984, 1987, DROUHARD, 1987) par rapport à l'activité mathématique (ce qu'il "faut" faire/ce qu'il "ne faut pas" faire) (PAQUELIER, 1986).

Dès lors, il n'est pas étonnant que le fait de prendre la parole, pour proposer ou défendre une solution, apparaisse à l'étudiant comme une activité vide de sens et dénuée d'enjeu, puisque c'est le maître qui prend en charge la question de la validité de cette solution.

2 Faire en sorte que l'étudiant parle du problème plutôt qu'il ait le sentiment que "le problème parle de lui" au travers du verdict de l'enseignant.

Très souvent l'étudiant aborde la recherche d'un problème avec une mentalité de "victime", victime d'un jeu dont les règles lui échappent et dont le dénouement sera son appréciation par l'enseignant. Dans cet état d'esprit, il s'agit donc pour l'étudiant de se "protéger", en ayant recours à des astuces, des recettes, des automatismes (cf. CHASTENET et al., 1987), qu'il applique dès qu'il croit identifier le danger (exemple : calculer le discriminant dès qu'il y a du second degré...).

Le déplacement que nous souhaitions favoriser visait donc l'acquisition d'une certaine "autonomie" par rapport au texte du problème : dire s'il est classique ou insolite, reconnaître, le cas échéant, s'il est ambigu, mal formulé, envisager des prolongements, des conjectures permettant de l'enrichir... En un mot, parler (mathématiquement, ou métamathématiquement) du problème, dans un discours recherchant ou exposant sa résolution.

3 Faire en sorte que les élèves s'engagent (à la première personne) dans une discussion contradictoire portant sur la vérité de leurs affirmations.

L'idée, paradoxale en apperence, qui est sous-jacente à ce troisième point, est que l'élève ne peut accéder au jugement de vérité (vrai/faux) tant qu'il reste à un niveau formel, tant qu'il n'a pas été intimement convaincu, à un moment donné, que ce qu'il prétendait était vrai (ou faux).

Autrement dit, la vérité d'un énoncé mathématique, qui typiquement ne dépend ni des circonstances, ni des individus, ne prend de sens pour l'élève que dans l'exacte mesure où elle a été, à un moment donné, "sa" vérité ("contextualisée" et personnalisée).

Cette personnalisation passe, à notre avis, par le débat contradictoire : lorsque l'élève prétend que tel énoncé est vrai tandis que son voisin lui soutient mordicus qu'il est faux.

En bref, les trois points évoqués ci-dessus concernent :

1 : le rapport de l'élève au maître.
2 : le rapport de l'élève au (texte du) savoir.
3 : le rapport de l'élève à ses condisciples.»

BIBLIOGRAPHIE

ALIBERT D. (1987) : Situation codidactique et délocalisation du savoir, *Publication annuelle du séminaire de didactique des mathématiques...*, U.S.T.M.G, Grenoble (à paraître).

BOURDIEU P., CHAMBOREDON J-Cl., PASSERON J-Cl. (1983) : *Le métier de sociologue*, Mouton (La Haye) et EHESS (Paris).

CHASTENET DE GERY J., DROUHARD J-Ph., HOCQUENGHEM S., et al. (1987) : Enquête sur les acquis ..., Actes du colloque *"Orientation et échecs ...*, Université de Paris-Dauphine.

GROUPE DE RECHERCHE SUR LA DIDACTIQUE DE L'ENSEIGNEMENT SUPERIEUR (ALIBERT GRENIER LEGRAND RICHARD), (1987) : Alteration of didactic contract in codidactic situation, *Proceedings of PME XI*, vol. III.

DROUHARD J-Ph (1987) : Approche linguistique d'une distinction intuitive : erreurs régulières vs. erreurs sporadiques en algèbre élémentaire, *Actes des Journées de la CIEAEM*, Sherbrooke, Québec.

DROUHARD J-Ph., PAQUELIER Y. (1987) : Recherche d'une démarche d'enseignement en mathématiques, avec des adultes, *Education permanente*,90 (ce texte, dans une version remaniée, forme le *Cahier de Didactique des Mathématiques*, 44, IREM Paris Sud).

GRENIER D., LEGRAND M., RICHARD F. (1985) : Une séquence d'enseignement sur l'intégrale en DEUG A 1ère année, *Cahier de didactique des mathématiques*,22, IREM Paris-Sud.

ALIBERT D., GRENIER D., LEGRAND M., RICHARD F. (1986): *Introduction du débat scientifique dans un cours de 1ère année du DEUG A ...*, Rapport ATP, U.S.T.M.G, Grenoble.

LACOMBE D. (1984) : Spécificités du langage mathématique et difficultés pédagogiques résultantes, *Actes des 6è J.E.S.*, GIORDAN et MARTINAND (eds.), APDRS, UER de Didactique, Université Paris 7.

LACOMBE D. (1987) : "Commentaire sur l'enquête précédente", Actes du colloque *"Orientation et échecs dans l'enseignement supérieur ...*, Université de Paris-Dauphine.

PAQUELIER Y. (1986):"Argumentation et situations didactiques : une démarche", *publication annuelle du séminaire de didactique des mathématiques et de l'informatique*, U.S.T.M.G, Grenoble.

ON HELPING STUDENTS CONSTRUCT THE CONCEPT OF QUANTIFICATION

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Abstract

This paper describes a genetic decomposition of the mathematical concept of quantification; that is, it gives a description of what could be the nature of a subject's understanding of this concept in terms of schemas consisting of objects and processes, and also suggests what specific reflective abstractions could be used in constructing it. The genetic decomposition is based on a general theory of knowledge and its acquisition, the researcher's mathematical understanding of quantification, and an analysis of protocols and other observations of students in the process of learning this concept.

We also discuss an approach to helping students learn quantification based on our theory and making use of computer experiences with the programming language ISETL.

Finally, we indicate the type of problems that students were given and the success that they had in solving them.

Introduction

Quantification of logical propositions over finite or infinite sets is a critical concept underlying a number of mathematical ideas from the elementary, such as the difference between an identity that holds for all values and an equation with particular solutions or the difference between a proof and a counterexample, to the advanced, such as the definition of the limit or the axioms for a group. Many students have serious difficulty in constructing this concept and this may help to explain their lack of success in understanding mathematics at several levels.

In the work described here, we have attempted to understand the psychological process of constructing quantification and use it to design instruction, especially using computers, that can foster the development of this concept by students. Thus there are two kinds of activity to be reported. The first is a theoretical analysis of the concept of quantification and how it is acquired. The second is a description of the instructional treatment along with the results obtained.

Part of this work was done jointly with F. Ellerman and C. Gong.

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Theoretical analysis

Our analysis of quantification is based on a general theory derived from Piaget's concept of reflective abstraction [3,5,9]. We have used it to study mathematical induction [4,7] as well as quantification [6] and we are presently working with it to study functions, sequences and limits (the last two with B. Cornu). According to this theory [5], mathematical knowledge and its acquisition is described in terms of *schemes* corresponding to specific mathematical concepts. A schema is a more or less coherent collection of (mental or physical) objects and internal processes which are applied to these objects. A schema is constructed by means of certain cognitive activities called *reflective abstractions*. These activities include *interiorization*, which is the construction of an internal process relative to a series of actions that can be performed or imagined to be performed on objects, *coordination*, which is the construction of a new internal process by combining two or more existing processes; *reversal*, which is the creation of a new process by inverting an existing process; *encapsulation*, which converts a process into an object by seeing it as a total entity, and *generalization*, in which an existing process is applied to an object for the first time.

Our theory hypothesizes that all objects and processes in logical thinking are constructed in this way, beginning with physical action schemas present at birth and continuing on up through the construction of new mathematics at the research level. Figure 1 describes our notion of schemas graphically.

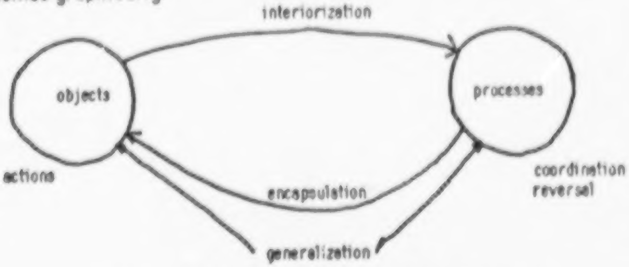

Figure 1. Schemas and their acquisition

A description of the schemas concerning a particular mathematical concept along with the reflective abstractions by which they may be constructed, is called a *genetic decomposition* of the concept. It is derived from three sources. The first is the general theory. The second is the mathematical knowledge that the researcher has about the concept. Third, the most important source is the information obtained from close observation of students in the process of learning the concept. The observations can take the form of clinical interviews, analysis of student errors in performing mathematical tasks related to the concept, etc. The researcher attempts to express the concept in terms of objects and processes. The next step is to look at the difficulties which the students are having and try to see if other students have overcome these difficulties by appearing to make particular reflective abstractions. The processes, objects, and reflective abstractions are then organized in a form that expresses the observed development of the concept by the students.

The resulting genetic decomposition is then used as a guide in designing instructional treatment. In our work we have relied heavily on setting the students to perform various tasks using computers because it seems to us that certain computer activities are very similar to the reflective abstractions and that students who engage in these activities may be more likely to make the reflective abstractions in their minds. As we proceed through the instruction, the same sort of observations are made continually and the genetic decomposition is revised accordingly. Thus at any point in time, what we use is only an approximate description of how some students may construct a concept along with computer activities designed to foster the constructions in that approximation.

We feel that this approach is very much in the same spirit as the *Teaching Experiment* of L. Steffe (as described in [8]) and the *Didactic Engineering* used in France [1]. Our genetic decomposition of quantification is the result of several years of theoretical analysis together with teaching experiments, extensive observations of students and analyses of these observations. Here, we concentrate on the resulting decomposition. Its relation with the theory and the mathematical notion should be clear. Space limitations prevent more than brief examples of how interview protocols influenced the genetic decomposition. Full details will appear elsewhere.

A genetic decomposition of quantification

The construction begins with cognitive objects that are simple declarations that may be true or false. These are made more complex in two ways: by linking several with the standard logical connectors (and, or, etc.); and by introducing variables to obtain proposition valued functions. In both cases these are actions on the objects -- the linking and the function action -- which must be interiorized to obtain processes.

The *single-level quantification* is constructed by coordinating these two processes to obtain the single process of iterating through the domain of a proposition valued function, checking the truth or falsity of the resulting proposition and applying at each step conjunction or disjunction according to whether it is a universal or existential quantification.

In order to move on to two-level *quantifications*, which are two (usually different type) quantifiers applied to a proposition valued function of two variables, the subject must encapsulate the above process to see that a single-level quantification has the effect of replacing the function by a single proposition. If the original proposition valued function involved two variables, the effect of this encapsulated single-level quantification is to replace it with a function involving one variable to which a second single-level quantification can then be applied. Thus the two-level quantification consists of parsing the original statement into two quantifications and then coordinating two applications of single-level quantification with an intervening encapsulation.

For three and higher level quantifications, the above procedure is iterated. When there are more than two quantifications, the parsing is non-trivial because there are several ways to group the statements. Since some of them make more sense than others, this provides one way to gauge the students' understanding.

Our observations usually begin with non-mathematical examples in order to minimize the difficulty. As the students develop their concept of quantification, mathematical content is gradually introduced. Also, our initial examples do not refer to familiar situations because we do not want the students to rely on remembered experiences, but rather to construct objects and processes that might be new for them and must be imagined.

For example, consider the protocols of three students who were asked what they would do to determine if the following statement is correct. They are taken from a course taught by the author at a western U.S. state university in the preliminary stages of the study.

> Amongst all the fish flying around the gymnasium, there is one for which there is, in every computer science class, a physics major who knows how much the fish weighs.

This statement requires, inter alia, the interiorization of three processes of iteration of a variable over its domain --- fish, classes and students. The following student response indicates an iteration of the fish, but his confusion suggests that it may be an action not yet internalized. It is not clear if he is iterating over students and the classes are ignored completely. Also, he does not succeed in applying quantifications.

> STUD1. I would collect all of the fish in the gym and if one of them (pause)... I get one fish... you know, I go through each of the fishes and then all of the computer science students know how much that fish weighs... for each of the fishes... no, for one of the fishes.

The second student does seem to have interiorized iterations over the classes and the students, but perhaps not the fish.

> STUD2. Okay, you take all this -- you take the set of fish that are in the gym and the set of students that are in computer science -- you would take the set of all the computer science classes and the set of all the students that are in those computer science classes and check to see if there was a student in one of those -- yeah, at least one student in every one of the classes that knew how much one of those fish in the gym weighed. And if that were true you would return true, and if it found one case where that failed it would be -- if there was one class with no students...
> INT. Okay, would you do it exactly the same as...
> STUD2. I'd probably go class by class and ask in each class if there was somebody who didn't know how much any of the fish in the gym weighed.
> INT. So you would have skipped the first set -- you skip the set of fishes?
> STUD2. Yeah. I think I'd probably try and prove it false, rather than trying to prove it true.

STUD2 applies quantifications to the two processes that she does seem to

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have, but cannot incorporate the fish, even with a prompt from the interviewer. This may be because she has not encapsulated her two-level quantification to see it as a single proposition valued function whose domain variable is fish. Compare this now with the following student who seems to have encapsulated the two-level quantification as a proposition valued function of the fish and then, with hesitation, seems to iterate the fish over its domain and possibly applies the final quantification.

> STUD3. Okay, I would look at a set of fish among the set of all available fishes and I would have to iterate over that and of course the condition is that as soon as I find the first one for which the rest of the long expression holds, I stop right then and there.
> INT. Can you explain to me what would be the rest of the whole expression? How would you check that?
> STUD3. Yeah, that was just the first step. I got...I'm picking a fish and then I have to start iterating over a set of available classes. Here I'll have to go through every one of them for that fish. And then I would have to go through a set of students in the class. Here we're dealing with an exists so that as soon as we find the first one that matches the rest of the conditions, its fine. And then I would run that function on the student.
> INT. What would you ask about the student?
> STUD3. I would ask if the student knows the weight of the fish.

The above discussion involves only the object proposition that results from a quantification. There are many other aspects to the genetic decomposition of quantification. These include a schema for negating a quantified proposition and the action of reasoning about a proposition. There is not space here to go into these matters so we refer to the paper which describes the analysis more completely [6].

Instructional treatment

The instructional treatment makes use of the programming language, ISETL which is an interactive language that supports most of the standard constructs of mathematics in standard mathematical notation. Otherwise, the programming syntax requirements are minimal and students spend almost all of their programming effort on issues of mathematics as opposed to programming per se. Beginners tend to have little difficulty learning the language and in a short time are able to construct fairly complicated mathematical objects and processes. The way in which the language is used is to ask the students to write programs chosen so that thinking about them will tend to induce the student to make the appropriate reflective abstractions necessary to construct the schemas for the concept of quantification.

For full details on the use of ISETL in this and other mathematical contexts, the reader is referred to [2]. In this paper, there is only space for some general illustrations. For example, the linking of declarations with conjunctions or disjunctions can be programmed directly and the student can write procedures to implement proposition valued functions. The quantifiers are invoked by using the keywords *exists* and *forall*. It is possible to write a procedure that will accept a proposition valued function and a finite subset of its domain and return *true* or *false* as the result of a quantification. The use of these procedures can be iterated to obtain higher-level quantifications. For each step of the genetic decomposition described above, one or more types of tasks with ISETL are used to help the student take the step. All of this treatment has been integrated into a full course in discrete mathematics [2].

Results

Again, there is only room for a representative sample of results. The following data is a selection from a single class of 19 students, average age about 20, taught by the author at a small private engineering and science university in upstate New York. This approach has been used by the author and several colleagues at a number of schools in the United States. The results seem to be generally consistent with what is reported here. It is necessary to view these results in light of the fact that, in the United States at least, students do not usually succeed in gaining much understanding of quantification. The 10 representative questions are grouped in four categories.

A. Express the following statement in formal language.

1. There is a year in the 19th Century during which in Potsdam it snowed at least one day in every month.
2. For every book in the library, there is a number of days (less than1000) such that if the book is that number of days overdue, then the fine is $10.
3. Same as fish statement used in the protocols.

B. Negate the following statement.

4. For every city in Vermont, there is a city in New York which has the

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same name.

5. There is a positive number Q such that for every positive P and for every x in $[c-Q,c+Q]$, if $|x-c|<Q$ then $|F(x)-F(c)|<P$.

6. Same as question 3.

C. Describe how you would determine if the following statement were true or false.

7. Same as question 1.

8. Same as question 5.

D. Reasoning about the fish statement used in the protocols.

9. What can you say if you know there are no computer science classes?

10. What can you say if there are computer science classes, but none has a physics major?

Here are the results for the 19 students.

Question:	1	2	3	4	5	6	7	8	9	10
Percent Correct:	69	47	84	95	55	71	75	76	82	82

Questions were either given for the students to do in class or as home assignments. Note that both relatively low scores (Questions 2 and 5) came with statements involving implication. The only other implication occurred in Question 8. on which the students did well. One possible explanation for the diffeence is that students can work through an implication but, at least in the presence of a quantification, have difficulty expressing or negating an implication. The results do not suggest any other difficulty.

References

1. Artigue, M. *Ingénierie didactique à propos d'équations diférentielles,* **Proceedings of PME XI**, Montreal, 1987.
2. Baxter, N. , E. Dubinsky & G. Levin, **Learning Discrete Mathematics with ISETL**, New York: Springer (in press).
3. Beth, E.W. & J. Piaget, **Mathematical Epistemology and Psychology** (W. Mays, trans.), Dordrecht: Reidel, 1966. (Original published 1965.)
4. Dubinsky, E. *Teaching mathematical induction I*, J. Math. Behavior, 5, 305-317 (1986).
5. Dubinsky, E. *Reflective abstraction in advanced mathematical thinking* (in preparation).
6. Dubinsky, E., F. Elterman & C. Gong, *The student's construction of quantification,* For the Learning of Mathematics (in press).
7. Dubinsky, E. & P. Lewin, *Reflective abstraction and mathematics education: the genetic decomposition of induction and compactness,* J. Math. Behavior, 5, 55-92 (1986).
8. von Glasersfeld, E. *Learning as a constructive activity* (reprint).
9. Piaget, J. *Piaget's theory,* in P.B. Neubauer (Ed.) **The process of child development** (pp. 164-212), New York: Jason Aronson, 1975.

CHILDREN'S LEARNING IN A TRANSFORMATION GEOMETRY MICROWORLD

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Research in mathematics education has in recent years focused on two important areas: the development of fine-grained models of learning, and the explication of principles and practices in the design of effective instructional environments. This paper discusses an ongoing research project which investigates in depth children's mathematics learning in a particular kind of instructional environment, a computer-based microworld. The domain chosen for the microworld is transformation geometry. This domain is both mathematically rich, and also connected with children's everyday experience with motions in space. The purpose of the research is to document the process of meaning-construction that takes place as middle-school students interact with the microworld, and to use this empirical data to engineer a more effective learning environment.

Instructional environments for mathematics created in the past decades have in many cases been characterized by a concern for making connections with children's experience. For example, the Logo environment ties computer operations to children's "personal geometry" (Papert, 1980); we also see an increasing use of manipulatives, for instance Cuisinaire rods, in mathematics instruction. Such manipulatives and concrete computer environments allow students to build models of mathematical concepts from the bottom up, basing their understanding on experiences under their own control (for examples of computer learning environments, see Schwartz, 1987 and Thompson, 1985). This paper describes an iterative, principled design for a curriculum in transformation geometry which is centered on an interactive graphical computer microworld. A microworld is an instantiation of the central objects and relations of a domain into a concrete form which is accessible to new learners. This microworld instantiates three Euclidean, or distance-preserving transformations: translation, rotation, and reflection, as well as scaling and shearing.

The domain of transformation geometry was chosen because it is both mathematically-rich and at the same time grounded in everyday experience with motion and imagery. Thus, students bring to the learning situation antecedent conceptual structures, strategies, and expectations. The primary research problem has been to investigate the interactions between the learners' prior knowledge, and the new domain of mathematical experience presented in the curriculum. Information about these interactions, including misconceptions, successful transitions, and conceptual change, is used to build a model of the typical "learning path" through the domain, and to modify the microworld and the curriculum.

OBJECTIVES

The objective is to create a plausible model of the way children construct meanings for the new mathematical entities in the microworld, and to use this model to refine and improve the pedagogy. A specific focus of the research is on the children's use of qualitative, visual or imagery-based strategies as well as quantitative and symbolic methods for problem-solving. The primary analytic frame is genetic task analysis (diSessa, 1982; Kliman, 1987), in which empirical observations are employed to create a model of the learner's changing conceptual structure. This paper presents preliminary results from two rounds of pilot-testing of the curriculum with fourteen-year-old students, as well as a description of questions to be pursued during expanded use of the microworld in the near future.

THE MICROWORLD

The microworld presents the learner with a computer screen, representing the plane, a grid showing the origin at the center of the screen, and a simple plane figure shaped like the letter L. Transformations available include: slide (translation), rotate, and reflect (all of which preserve distance); scale (which changes size but preserves shape); and shear (which changes shape but preserves area). There are also simple, "local" versions of several transformations, namely, pivot, flip, and size. These simple versions transform the shape "in place," rather than using the whole screen, and are intended to correspond to children's initial conceptualizations of these operations.

One functional activity programmed into the microworld is the Match game. The user is asked to enter a sequence of transformations to move the center shape until it is superimposed onto a congruent shape placed randomly on the screen. The purpose is to give students experience with the individual transformations, and with decomposing a complex mapping into simpler operations. It is also expected that students will begin to understand and generalize about the properties of the various transformations, as they use them strategically to achieve the goals of the game.

METHODOLOGY

There have been two rounds of pilot-testing of the microworld, with the final data collection to take place in a middle-school this term. In the initial field test, 65 seventh and eighth grade students (age range twelve to fourteen) used the microworld in pairs. Half the students had completed a two-week unit in their mathematics class on transformation geometry, and the other group were novices. Both groups enjoyed using the microworld, particularly the Match game. The two groups differed somewhat in their strategies for the game. The group with previous domain experience tended to use the whole-plane versions of "rotate" and "reflect," while the novices used the simpler "pivot" and "flip." Several students discovered a fool-proof game strategy which involved sliding then pivoting and/or flipping the shape.

In the second round of piloting three students were videotaped using the microworld with the investigator. None of these subjects had previous experience with transformation geometry, although two (Charles and Joanna) had Logo experience.

THE TASKS

Joanna, Charles, and Lee spent from 1 1/2 to 3 hours using the microworld and doing paper-and-pencil tasks. The sessions included time for free exploration and playing the Match game, as well as for simple problem-solving. The tasks included:

1. Identification of transformations:
Five transformations were illustrated on paper, and the student was asked to name each and give the correct Logo command.

2. Execution of transformations:

The student was given six sheets showing the starting figure and a Logo command, and was asked to sketch the result.

3. Finding inverses:

Working on the computer with the investigator, the student was asked to find the transformation which would "undo" an operation.

4. Combining transformations (composition):

Student was asked to predict the effect of two transformations performed sequentially. Then he/she was asked to find a single equivalent transformation.

RESULTS

1. Identification task:

The students were least successful on this task. Out of the 14 attempts (one of the 5 problems was not given to Joanna), 9 were eventually correct. Four of the correct responses were arrived at following multiple attempts, using either sheets of tracing paper or the computer for trial and error. The most difficult of the identification tasks was finding the center point of a rotation. Out of six attempts, three were successful.

There were two interesting strategies used in the unsuccessful responses. Joanna, when trying to find the center point of a rotation, tended to select the midpoint of an imaginary line connecting the starting vertices of the two shapes, as illustrated in Figure 1. In choosing this point, the subject seemed to be ignoring some of the constraints of the problem - her "center point" would map the two vertices onto each other under a 180° rotation, but the rest of the figures would not match. She also did not seem to see that the figures differed in heading by 90°, not 180°. She did not use tracing paper or any other methods to check her answer, and when I asked her, "Is there any way you can figure it out?," she justified her answer by saying "Well, because there's an equal distance between there [from the "center point" to each vertex], so it could use this point and turn around that." This response suggests that the midpoint chosen by Joanna was visually salient, but distracted her from seeking the correct center point. It also highlights the utility of having concrete or other methods to check one's answer.

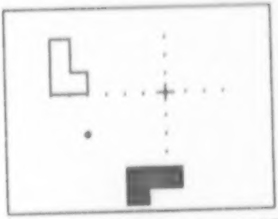

Figure 2: Joanna's Incorrect Center of Rotation

A second strategy used in the rotation tasks reflects what I believe is a naive conception about transformations. Several students, including Lee, interpreted a rotation not as choosing an arbitrary point and then turning the whole plane around it, but instead as a composite motion which moves the shape to a specified point and then turns it around its starting vertex. This misinterpretation of rotation is consistent with a conceptualization of transformations which does not think of the plane as the object being transformed. Instead, it is the figure which is being moved about or changed, with the grid being used to say how much or where. Some students found imagining a rotation around a point not on or in the figure itself to be difficult. This might be because of previous experiences with rotations, such as record turntables, door handles or the hands of a clock, in which the center of rotation is always within the bounds of the object. A "bridging activity" was used by the investigator to help students who had this "local" conceptualization of rotation. The students were asked to think of a string or stick between the figure and the center point of the rotation, and then to imagine rotating the string plus figure through the specified angle. A more concrete version of the activity was to have the student perform the rotation with tracing paper, first "pinning down" the center point with a pencil. Both of these activities allowed the child to think of the figure plus string, or figure on paper, as one object, and thus were consistent with prior conceptions.

A possible modification of the microworld would address the more general issue of the students' awareness of the plane as the object undergoing a transformation. The plane may not be inferred or constructed as a conceptual object by the students because they see only the "motion" of one shape, and so the idea of the plane as a span of infinite points or locations may not occur to them. Adding another object to the screen, which is mapped under the same transformation, should allow the student to realize that both figures, and any other set of points, are part of a larger plane, and the transformation applies to all the points in that plane. This modification will be tested when the microworld is used with the next group of children.

2. Execution task:

The students in the pilot group were much more successful at the execution task. Sixteen out of the seventeen solutions were correct. Strategies used included visual estimation, paper folding, rotating and tracing, and use of rulers.

3. Finding inverses:

The investigator worked individually with the students for this task and for combining transformations, and the specific problems varied from subject to subject. The students were all successful at predicting the operations which would "undo" various transformations. One interesting error occurred when Charles spontaneously tried to find the inverse of a slide. He entered "Slide -120 -123" and then said "I'm seeing if I can get it back." His first try was "Slide -123 -120." That is, he reversed the order of the inputs. When this did not have the desired effect, he thought a bit more and realized he would have to change the signs of his original inputs ("I have to have it positive"). His first impulse is interesting, though, because it suggests a kind of shallow "symbol-pushing" of the numbers, rather than a precise understanding of how the Logo commands are used to represent motions of the plane. Charles corrected himself by making use of the visual feedback from the microworld, and other students were observed trying various inputs to commands and watching the results in order to disambiguate their understandings of the operations.

4. Combining transformations:

Again, the students were in general successful in performing such tasks as predicting the outcome of two slides, and finding a single expression with the same effect. The most difficult combination was two reflections in parallel axes. The outcome is a slide, and though it is not expected that students will come up with an algebraic formula to find the displacement, they can read it off the screen. Joanna and Lee used the screen information to do so. Charles once again seemed to focus on modifying the Logo commands, at first adding the parameters of the two reflects and then trying to take the average. After several tries, he noticed with a little surprise that two reflects don't make a reflect, but instead a slide, and found the correct parameters.

DISCUSSION

The preliminary results from the piloting highlighted a number of issues which will be investigated further in the next phase of the research. This phase will consist of expanded use of the microworld with a larger number of students. The topic of transformation geometry will be introduced to a 7th grade class at a local middle school, using paper folding and tracing, the microworld, and class discussion. Five pairs of students, reflecting a range of mathematical ability, will be selected to continue working with the microworld for about six more sessions, which will be videotaped. In addition to the think-aloud protocol data, the students will also be given paper-and-pencil measures, such as an adaptation of the transformation geometry section of Hart's mathematics tests (Hart, 1981).

The issues to be addressed in this phase include the students' understanding of transformations as whole-plane operations, and as conceptual objects in themselves. That is, it may be that students start out by seeing the transformations simply as ways to get the shape to behave as they desire. The inverse-finding and composition tasks are intended to encourage the students to consider the transformations as objects which can be combined, and to see that certain combinations form systems with interesting properties. To further this aim, a new task will involve selected sets of transformations which make up groups, for example, groups of rotations around a single point, or the reflections and rotations of a square. The pairs of students will be asked to explore these special sets of transformations, by answering such questions as: How many ways can you transform a square so that it stays in the same place? What happens when you perform two reflections in perpendicular axes? and so on. In using the microworld to answer these questions, the students will be encouraged to look for patterns, to generalize, to test hypotheses on different examples, and to document the results of their explorations in written form .

A second new task will involve using the transformations to generate or match symmetric patterns, such as wallpaper designs. This activity gives the students more scope to use the transformations for projects of their own choosing, and should provide information on how they think of the transformations once they are beyond the initial learning phase.

The results presented here provide the first sketchy elements of a more complete learning path through the domain. The computer microworld gives the students an arena in which to explore the transformations, starting with single operations and working towards a non-formal understanding of groups of transformations. It also provides the researcher with a modifiable tool for probing the learner's changing conceptualizations. The next phase of the research will continue the iterative process of building a model of the learning and using this information to improve the learning environment.

REFERENCES

diSessa, A. (1982). Unlearning Aristotelian physics: a study of knowledge-based learning. Cognitive Science 6. 37-75.

Hart, K. (1981). Children's understanding of mathematics: 11-16. London: CSMS, University of London.

Kliman, M. (1987). Children's learning about the balance scale. Instructional Science 15 (4). pp. 307-340.

Meserve, B. (1955). Fundamental concepts of geometry. New York: Dover.

Papert, S. (1980). Mindstorms: children, computers, and powerful ideas. New York: Basic Books.

Schwartz, J. & Yerushalmy, M. (1987). The Geometric Supposer: an intellectual prosthesis for making conjectures. The College Mathematics Journal, 18 (1).

Thompson, P. (1985). Experience, problem solving and learning mathematics: Considerations in developing mathematics curricula, in E. Silver (ed.), Teaching and learning mathematical problem solving: multiple research perspectives. Hillsdale, N.J.: Lawrence Erlbaum Assoc.

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SOME COGNITIVE PREFERENCE STYLES IN STUDYING MATHEMATICS

Dr. HAMDY EL-FARAMAWY

Abstract

The present study was sorted to explore the cognitive styles concerning the study of mathematics. Cognitive styles were focused are Field-independence-dependence and impulsivity-reflectivity.

Samples was chosen from College Students in Menoufia University-Egypt, they devided into two groups according to their achievement in mathematics. EFT and MFFT were used to assess the student's cognitive styles. Results indicated that high achievement students in mathematics are more field independent and more reflective than low achievement students.

Introduction

It became clear that students of approximately the same aptitudes do not always perform equally well in the classroom. The researches in that field showed that the style which student brings it to a task treating with the information, might be behind that differences in performance. The style which person prefer to deal with information or response called "cognitive style". Witkin _et al_. (1977) define cognitive style as follows: "...... the characteristics approach the person brings with him to a wide range of situations - we called it his "style", and because the approach encompasses both his perceptual and intellectual activities - we spoke of it as his "cognitive style" (P. 10).

Cognitive style develops in the form of cognitive structures which become more differentiated with time and experience and which become increasingly stable as they devlop[*]. Also, it could be said that the cognitive styles indicate the form of cognitive activity not its content

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[*] This point was treated by the present researcher in paper (in arabic) titeled "Cognitive styles and psychological differentiation-Theoritical study" (in) The Second Annual Conference of Egyptian Psychological Society, 1986.

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It has been established by researchers that the cognitive style is not as the ability. As Messick (1976) argues that abilities and cognitive style are linked by the concept of performance, since ability implies the measurement of capacities in terms of maximum performance, while cognitive style implies the measurement of characteristic modes of operation in terms of typical performance. Further, he sees that although abilities are unipolar while cognitive styles are bi-polar both range from one extreme to opposite extreme.

In many instances the investigators claim to have discovered bi-polar dimensions, and Messick (1976) lists these dimensions together with other cognitive styles categories, as follows:

- Field independence vs. Field dependence.
- Field articulation.
- Conceptualizing styles.
- Breads of categorization.
- Conceptual differentiation.
- Compartmentalization.
- Conceptual articulation.
- Integrative complexity.
- Cognitive complexity vs. Simplicity.
- Leveling vs. Sharpening.
- Scanning.
- Reflection vs. Impulsivity.
  Risk taking vs. Cauti ousness.
- Tolerance for unrealistic experience.
- Constricted vs. Flexible control.
- Strong vs. Weak automatization.
- Conceptual vs. Perceptual-motor dominance.
- Sensory modality performances.
- Converging vs. Diverging.

In spite of the claims made by the respective authors for their dimensions, there does remain a big confusion surrounding so many of them, however some of them became most clear or more determined like

"Field dependence vs. Field independence", and "Impulsivity vs. Reflectivity", which the present study is concerning.

Furthermore, the researches indicated that cognitive styles has a potentially important relationship with educational performance generally, and a student's academic choice especially.

Statement of the problem:

If cognitive style is the mode of organizing or categorizing the environment, therefore, it must be manifest as a factor in school learning. However, there are acumulated researches related that field, attention should be given to more systematic researches trying to investigate the cognitive styles influence in learning each of subject matter in which lead to high achievement. Such researches enables us to establish map of cognitive styles represent a picture of the variety of profiles the student uses in his education. Then we could translate educational and psychological research on cognitive styles into practice. Accordingly, the teacher ables to consider the individual in terms of this map, and he could match the student to appropriate task. The present study however, trying to demonistrates cognitive styles which should be used by students studying mathematics to achieve a high level in that field.

Previous attempts:

There are a lot of researchs which examined the relationship between academic achievement and cognitive style. Haskins and Mckinney (1976) found concurrent relationships between the performance on matching familiar figures test (MFFT) and IOWA achievement test scores in elementary school children, but they did not explore information about the predictive capacity of the MFF with respect to academic achievement. Also, the differences in achievement between reflectives and impulsives-children identified as reflectives in grade 4 had significantly higher achievement test scores in grade 5 and 6 those identified as impulsives (Barrett, 1977).

Moreover, there are a number of researchs examined this relationship with different cognitive styles and other educational fields such as, Tamir (1976) who examined the relationship between achievement in biology and cognitive preference styles (Analytical, Relational and Inferential).

In high school students with results that show that levels of cognitive style and achievement are related to four independent variables, namely sex, school environment, the nature of the curriculum and the attitudes of teachers toward in inquiry-oriented curriculum. Similarly, McNaught (1982) has studied the same cognitive styles of secondary school chemistry student and he established a significant correlation between these cognitive styles and differential achievement in particular tasks in an achievement examination.

Recently, however, attention has been given to matching between teacher's cognitive styles and student's achievement. For example, Saracho and Dayton (1980) have examined this relationship for 2nd and 5th graders, 36 teachers and 132 children were administered the embedded Figures Test (EFT) to measure field-independence versus field-dependence cognitive style. Children and teachers with similar cognitive styles were considered matched, whereas children and teachers with different cognitive styles were considered mismatched. Results indicated significant effects on achievement due to teacher's cognitive styles, but there was no significant outcome associated with the matching variable. However, children with field-independent teachers irrespective of their own styles showed greater achievement gains than children with field dependent teachers. Also, the present researcher (El-Faramawy, 1984) explored the relationship between students' and teachers' cognitive styles, and the influence of this relationship upon student's academic achievement in biology and academic tendency toward the subject matter. The dimension of cognitive style in which the study focused was kagan's "Inpulsivity-Reflectivity" cognitive style. Results indicated that level of student achievement appears to be related most closely to the variable of teacher cognitive style, however the correspondence in the case of impulsivity does not appear to increase significantly expectations of high student achievement but may contribute to medium achievement and medium tendency.

Definition of terms:

It is essential at the out set that the basic terms used in the study are clearly defined:

Cognitive styles:

The present study would be concern two cognitive styles called impulsivity versus reflectivity and field dependence versus field independence.

The firt style was defined as the tendency to reflect over alternative solution possibilities (reflective) in con trast to the tendency to make an impulsive selection of a solution in problems with high response uncertainty (impulsive) (Kagan, Rosman, Day and Phillips,1964).

The style operationally, is defined by Kaggan and his associates (1964) as two dimensions, namely latency (times) to first response and accuracy of choice or total errors. These two dimensions are both assessed by the Matching Familiar Figures Test (MFFT). This style would be assessed also in the present study by MFFT which established by the present researcher.

However, the other style called Field dependence versus field independence was conceptualized by Witkin et al. (1962) as the tendency to perceive and restructure a stimulus field in order to separate items from that field and view the field in "parts" field-independent people are analytical in their perceptions of a field and can separate out discrete objects. Field-dependent people perceive a field in more global fashion and are less likely to dis embed the discrete objects within the field. This style would be assessed in the present study by Embedded Figures Test (EFT).

Hypothesis:

The present study however trying to realize the following hypothesis. There are significant differences between student's performance

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in cognitive styles tests and their achievement levels in mathematics, as follows:

a) High achievement students are more reflective than low achievement students.

b) High achievement students are field-independent than low achievement students in mathematics.

Procedures and results

Sample chosen in the present study are 325 students (male and female) representing the fourth grade in the College of Education-Menoufia University, Egypt. Their mean age is 21.5 years.

According to mean achievement in mathematics of each student in his last year, two groups were chosen as follows:

- 27% of the whole sample (n = 88) achieved higher level in mathematics.
- 27% of the whole sample (n = 88) achieved lower level in mathematics.

Table (1) shows the two groups by mean of achievement and t value.

Table (1): Classification of groups by mean of achievement, standard deviation and t value.

Sample	Achievement			
	n.	m.	SD	t value
Higher Ach. 27%	88	9.06	1.16	25.48**
Lower Ach. 27%	88	5.63	0.48	

** = Significant at 0.01 level.

The findings in Table (1) indicate that there is difference in achievement between the two groups where higher achievement group has mean reachs to 9.06 and lower achievement group 5.63 and that difference is significant at 0.01 level (t value = 25.48).

Thereafter, the two groups were administered the "Embedded Figures Test" (EFT) and "The Matching Familiar Figures Test" (MFFT) to measure the two cognitive styles (Field dependence-independence and Impulsivity-Reflectivity, respectively).

The following tables shows results:

Table (2): The two groups by mean of independence, standard deviation, and t value.

Groups	(EFT) independence			
	n.	m.	SD	t. value
Higher Ach.	88	10.16	4.38	5.48**
Lower Ach.	88	6.58	4.24	

** = Significant at 0.01 level.

Table (3): The two groups by mean of latency, standard deviation and t. value.

Groups	MFFT latency			
	n.	m.	SD	t. value
Higher Ach.	88	983.09	337.05	2.29*
Lower Ach.	88	862.52	357.45	

* = Significant at 0.05 level

Table (4): The two groups by mean of errors, standard deviation and t. value.

Groups	MFFT errors			
	n.	m.	SD	t. value
Higher Ach.	88	21.16	11.03	3.5**
Lower Ach.	88	28.81	17.18	

** = Significant at 0.01 level.

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Discussion

The present study was designed to find out the cognitive styles contributing to a high achievement in mathematics. The findings emerging from the study reveal that the hypothesis of the study is supported. Overall, there are differences between high achievement students and low achievement students in their performance on cognitive styles' tests as we see in Tables 2, 3 and 4.

Table (2) indicates that higher achievement students are more independent than lower achievement students (mean independence = 10.16, 6.58 respectively) while t value between the two groups is 5.48 which significant at 0.01 level.

That finding is relating the cognitive style called "Field dependence vs. Field independence". However, the findings in Tables 3 and 4 are relating Impulsivity-Reflectivity cognitive style which is assessed in terms of two dimensions, they are latency and errors, the findings of latency and errors indicate that high achievement students are more reflective than lower achievement students. Tables 3 and 4 show that mean latencis = 983.09, 862.52 respectively and mean errors = 21.16, 28.81 respectively, where the two t values are significant at 0.05 and 0.01 level for latency and errors respectively.

Overall, it could be said that high achievement students are more field independent and reflective than low achievement students in mathematics.

The present study, however, suggests that if students studying mathematics are field independent and reflectives, then, they would be high achievers in mathematics. These result is expected because the field of mathematics (Applied and pure mathematics) requires analytical and geometrize treatment with the content of that field.

Also, this result suggests that much further studies are required to explore other cognitive styles influencing the field of mathematics and other fields of education, such researches enables us to detect possible trend data concerning cognitive styles mapping to be benifit in matching student with the suitable field of education.

References

Barrett, D.E. (1977). Reflection-Impulsivity as a predictor of children's academic achievement. Child Develop., 48, 1443-1447.

El-Faramawy, H.A. (1984). The correspondance between teacher and student cognitive style and its implications for academic achievement and academic tendency. Ph.D. Thesis, University of Wales, U.K.

Haskins, R. and McKinney, J.D. (1976). Relative effects of response tempo and accuracy on problem solving and academic achievement. Child Develop. 47, 690-696.

Kagan, J.; Rosman, B.L.; Day, D.; Alpert, J. and Phillips, W. (1964). Information processing in the child-significance of analytic and reflective attitudes. Psychological Monographs : General and Applied, 78(1) (Whole No. 578), 1-37.

McNaught, C. (1982). Relationship between cognitive performances and achievement in chemistry. J. of Research in Science Teaching, 10, (2): 77-86.

Messick, S. (1976). Personality consistencies in cognition and creativity. In Messick, S. & Associates (ed). Individuality in learning. Washington: Jossey-Bass Publishers.

Saracho, N. and Dayton, C. (1980). Relationship of teachers' cognitive styles to pupils' academic achievement gains. J. of Educ. Psychol., 72(4), 544-549.

Witkin, H.A. et al. (1977). Field-dependent and Field-independent cognitive styles and their educational implications. Review of Educational Research, 47, 1-64.

Witkin, H.A. et al. (1962). Psychological differentiation, New York, Wiley.

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EXPLORING CHILDREN'S PERCEPTION OF MATHEMATICS THROUGH LETTERS AND PROBLEMS WRITTEN BY CHILDREN

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Frequently, a direct questionnaire approach is used to explore the affective domain. In this study, however, an indirect task was used to explore children's perception of mathematics. A total of 94 Grade 6 children were asked to write a letter to a friend (who had been ill) and explain what mathematics had been covered while the friend was away from school. Through their responses to this task, children revealed their perception of the mathematics they had recently encountered at school. The results show that the teaching style encountered by the children is reflected in their letters. The lack of detail in many of the explanations is interpreted as either a lack of understanding on the children's part or as closely linked with the children's low self-esteem and interest in mathematics.

INTRODUCTION

Children's perception of mathematics is influenced by interaction with others - parents, peers and teachers - in the home, the playground and the classroom.

Just as children are engaged in constructing mathematical meaning from the sets of experiences and interactions that confront them (Bauersfeld, 1980; Labinowicz, 1985; Cobb, 1986), so they are constructing and establishing mathematical perceptions in the affective domain. As McLeod (1987a) pointed out, new approaches to research in the affective area are needed, and he suggested that techniques used in constructivist research in the cognitive domain could be applied in affective studies.

Increased emphasis on children's use of language in the mathematics classroom has paralleled the recognition of constructivist approaches in mathematics education. For example, Geeslin (1977) used writing about mathematics as a teaching technique, while Kennedy (1985) introduced several forms of writing to his mathematics students, including writing letters about what they were studying, keeping regular logs and devising mathematics problems about a specific topic. Ellerton (1980, 1986a,b) and van den Brink (1985) have introduced creative writing in mathematics by asking children to make up mathematics problems while Mett (1987) has used journal writing, writing in class and project writing as learning devices in calculus.

Children's expression of mathematical ideas through the creation of their own mathematics problems demonstrates not only their understanding and level of concept development, but also reflects their perception about the nature of mathematics (Ellerton, 1986b). In contrast, a direct questionnaire approach tests responses to researchers' pre-formed notions about the affective domain (McLeod, 1987a), and there is little, if any, opportunity for students to express their own ideas about mathematics.

As individuals, we reveal our attitudes towards a particular area of human endeavour through our facial expression, our body language, what we say, what we write and what we do. These *actions* reflect our *thoughts* and our *feelings*. Some tasks provide more useful mirrors for our thoughts and feelings than others. Actions that are well rehearsed (involving an automatic response, for example) will elicit less affective response than actions which stem from novel tasks or situations that need interpretation before they can be translated into action. This is consistent with Mandler's (1984) analysis of mind and emotion which interprets affective responses as arising mainly from the interruption of plans or of planned actions.

Thus observing a student who has been asked to solve a page of symbolic problems will provide limited affective data if this is the type of activity the student encounters every day. However, by providing a task which is more open-ended, and by allowing students to apply their own interpretation and emphasis within the task, students reveal more of their perception of mathematics and of their attitudes towards the subject.

WRITING LETTERS ABOUT RECENTLY STUDIED MATHEMATICS

In the study reported here, the following task was given to 94 children who were in their final year of primary school (Grade 6) and who were all about 12 years old. The children were in four classes in a school set in a low socio-economic suburb of the provincial city of Geelong.

> Imagine that you have a friend who has been ill and has missed about 3 weeks of school. Your friend has sent a message to you, asking you to explain what the class has been doing in mathematics so that he or she can do some extra work at home to catch up.
>
> Write a letter to your friend describing what mathematics you've been doing in class. In the letter, don't forget to give examples of some of the mathematics questions that your friend would need to be able to answer so that he or she could catch up. Make sure you explain how to answer these questions.

Table 1 summarises the children's responses. Some children described only one mathematical topic; others described several areas. Each topic or area listed by each child has been recorded in the table. Thus the number of topics listed by the children will equal or exceed the number of children. Brief background details concerning the four mathematics classrooms used in this study are given in Table 2. Note that the teachers were not present at any time either while the children were given the task or while they were working on it.

Table 1: Mathematics topics described by children in this study

	Tables	Four Processes	Long Division	Fractions	Decimals	Problem Solving	Percentages	Metric System	Area	Graphs	Money, Bartering	Average no. of topics per child
Miss C	1	2	15				3		2		1	1.1
Mr E	4	9			8	6		3		13	4	2.2
Mr L		7		25					2			1.2
Mr U	2	12		8	2							1.0

Table 2: Background details of mathematics classrooms

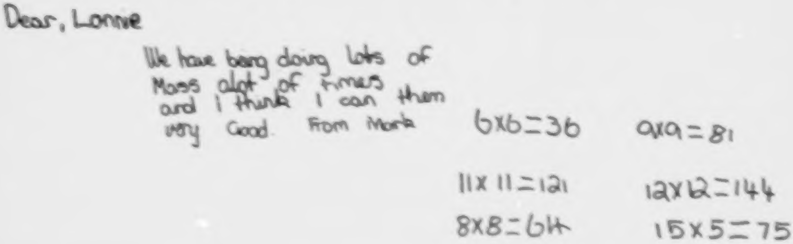
	Time on maths each week (hrs)	Fixed time slot	Materials used	Calculators	Computers	Group Activities	Yrs teaching experience	Most recent topics covered	No. of children in class
Miss C	6	✓	rarely	often	no	top group works with parent	5	long division	21
Mr E	6-7	varies	often	sometimes	$\frac{1}{2}$ hr/wk	often	13	graphs	21
Mr L	6	✓	sometimes	rarely	no	sometimes	14	fractions	28
Mr U	$4\frac{1}{2}$	✓	sometimes	sometimes	$\frac{1}{2}$ hr/wk	sometimes	15	revision topics	24

Of the four teachers listed, Mr E had intentionally tried to integrate mathematics across all areas of the curriculum. When he saw the write-a-letter task, he felt uncertain about whether the children would be able to *identify* mathematics from the contexts in which he had presented different topics. Pie graphs, for example, had been introduced in geographic and expenditure settings. Some children in this class initially responded to the write-a-letter task by telling me 'We don't do maths', to which I responded 'Just think about what maths you've done'.

The most striking aspect of the data presented in Table 1 is the spread of mathematical examples given by children in Mr E's class in contrast with the narrower focus of responses by children in the other classes. This suggests that, not only *were* the children able to identify as mathematics the topics presented in other areas of the curriculum, but they were *broader* in their interpretation of the task than their peers in the other three classes. They did not restrict themselves to only one specialised aspect of mathematics. Most children in Mr E's class identified at least 2 topics, and only four children listed just one area. This contrasts strongly with Mr U's class in which each child described only one mathematical topic in the letter.

Except for two children, all those in Miss C's class described either long division, area or percentages as one of their topics. One of the two exceptions described tables practice and this could be interpreted as being associated with estimations and calculations for long division. The other exception was a boy who listed 'addition sums' as the topic. His achievement on other mathematical tasks was low, and may indicate a reluctance on his part to describe long division, a topic in which he had little confidence.

Similarly, in Mr L's class, only three children did not include fractions as one of their topics. The exceptions (2 boys and a girl) achieved very low success rates in simple mathematical tasks, and found it difficult to express themselves in their letters. The girl, for example, wrote: 'We have had some maths but not much maths. We have been doing sheet nearly all the time'. She continued by (cleverly) avoiding a description of the maths on the sheet by writing: 'I will ask Mr L. if he could get some of these sheet and I will give them to you on Friday'. Both boys gave multiplication examples. The letter Mark wrote is reproduced below.



Dear, Lonnie

We have beng doing lots of
Moss alot of times
and I think I can them
very Good. From Mark

6x6=36
9x9=81
11x11=121
12x12=144
8x8=64
15x5=75

Again, one gets the impression that the children were being careful to keep their letters within the limits dictated by their own skills.

Mr U. placed less emphasis on mathematics than the other teachers, spending about two hours less per week on the subject than his colleagues. He had been revising different areas over the weeks prior to the task described here, and the children's responses were consistent with this. In general, the letters written by the children in Mr U's class were very brief and contained less descriptive detail and more symbolic notation than those written by children in the other three classes. On the other hand, letters written by children in Mr E's class contained *more* descriptive detail and less formal algorithmic process description than those composed by other children. The letters written by Miss C's and Mr L's children concentrated on describing the processes involved in the calculations.

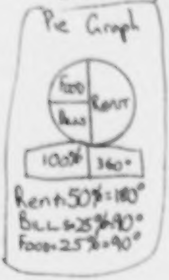
CHILDREN'S LETTERS REFLECT CLASSROOM EXPERIENCES

The children's responses have echoed their experiences in the mathematics classroom. Where algorithmic detail has been emphasised, this is reflected in the children's responses (Miss C's class), and where mathematics has been consistently presented as short episodes on one topic, this, too, is mirrored (Mr U's class). In sharp contrast are the broader mathematical experiences of children in Mr E's class in which mathematics could come into the discussion in any subject area. The children's letters mirror this; they are more general and cover a broad topic range. Mr E's children have perceived mathematics to belong to more broadly defined areas, with no single topic standing out in importance for attention in the letter, although graphical representations (which had been encountered most recently) were referred to by more than half the children.

SOME EXAMPLES OF LETTERS WRITTEN BY THE CHILDREN

From Mr E's class

To Watsname,
How are you feeling. We have been done graphs and 20 questions on Addition, subtraction, Multiplication and division. for example



Questions
1-20
365+2125
5450−162
212×65
5431÷12

From,
Watzmyname.
P.S. write again.

To frend ~~th~~ there have been a few
thing that you've missed out on like these
problems for you to work out.

5416
− 690

30.45
× 88·6

and we have been

dong graphs of desimals, cm, m, km, g, kg, t, l, ml
and alot of written ~~of~~ out bec problems
that we had to change into numerals.
in Sums

from Karen

From Mr L's class

B To Sandy

Im going very good at my mathematics at school but it is a little harth we have been doing Fa[illegible]s at school I [illegible] will show you one

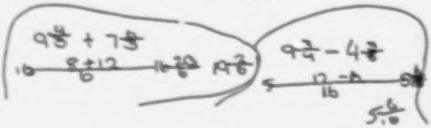
$3\,\frac{7}{1}\,\frac{17}{7}$

$6\,\frac{12 \div 2}{2}\;6\frac{2}{2}$

To Bradley

I am writing to you so you can find out what kind of maths we have [illegible] doing for the past three week we have done a lot of mostly fractions like take away, plus, and times once you get used to them they are easy to do look down below for examples by

from Jason

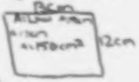


$9\frac{4}{5} + 7\frac{5}{5}$
$16\ \frac{8+12}{6}\ 16\frac{20}{6}\ 19\frac{2}{6}$
$9\frac{3}{4} - 4\frac{7}{8}$
$5\ \frac{12-8}{16}$
$5\frac{6}{16}$

From Miss C's class

Dear Friend,

The work we have been doing is Area. For Area you have to use the formula of Length×Width. here is an example



13cm
A=L×W
=13cm
×12cm
=156cm²
12cm

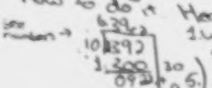
And when you have your answer you put a 2 after it The 2 stands for squared So if your answer was 156cm you would write 156cm².

Yours faithfully

Travis

Dear friend,

In mathematics lately, we've been doing long division, it's pretty easy once you know how to do it. Here's an example of how you do it;



see number →
39 r2
10)392
300
092
90
002
30
9
39
5.

1. What you do is guess a number, and times it by ten, or what ever number you are dividing by. My guess was 30, which I multiplied by 10.
2. Then I took it away from 392, which brought me to 92.
3. I then multiplied 10 by 9 and got 90.
4. Then I took 90 from 92, which equals 2
5. At last, I ~~[illegible]~~ added 30 and 9 which is 39, and the 2 is my remainder.
6. Finally, the answer is 39 r2.

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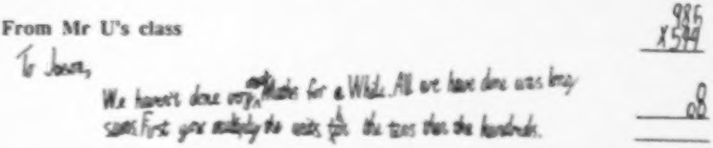
From Mr U's class



To Jason,

We haven't done very much Maths for a While. All we have done was long sums First you multiply the units then the tens then the hundreds.

986
X584

8
08

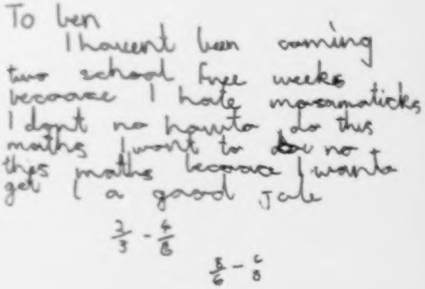
A USEFUL TOOL FOR STUDYING CHILDREN'S PERCEPTION OF MATHEMATICS

Providing children with a situation in which they need to describe recently encountered mathematics achieved the following:

1. It de-emphasised individual children and helped them to focus on someone other than themselves. It was then not obvious to the children that their responses could reveal personal attitudes and perceptions. (The question did not simply ask 'What mathematics have you been doing recently?')
2. The task forced several decisions such as: *What is mathematics? What mathematics have we done recently? Can I describe it? How should I describe it? What does my friend need to know? What can I leave out?* In Mandler's (1984) terms, the write-a-letter task is an interruption which will arouse the individual's nervous system. This response will make itself apparent as surprise, frustration, enjoyment or the like, and this, in turn, will be reflected in the type of written response given by the children.

A brief letter with no detail may imply little enthusiasm for mathematics (or poor understanding), for example. Several children used the opportunity to *boost* their self-esteem by saying that these problems were easy, or, like Mark in the letter to Lonnie, 'I think I can them good'. Generally, the children controlled their own emotions sufficiently to keep them implicit in their responses rather than explicit. However, an example is reproduced here in which the child's emotions were stirred to the point of becoming dominant in the letter.

The letter to Ben is really a cry for help from a child who is finding mathematics very difficult. This task proved to be a powerful way of tapping his feelings about mathematics, and helps to establish that this indirect method provides a valuable way of gaining access to the affective domain.



To ben
I havent ben coming
two school free weeks
because I hate mathamaticks
I dont no howto do this
maths I wont to do no
this maths because I wanta
get a good Jobe

$\frac{2}{3} - \frac{4}{6}$

$\frac{5}{6} - \frac{6}{8}$

REFERENCES

Bauersfeld, H. (1980). Hidden dimensions in the so-called reality of a mathematics classroom.*Educational Studies in Mathematics 11*, 23-41.

Cobb, P. (1986). Making mathematics: Children's learning and constructivist tradition. *Harvard Educational Review 56(3)*, 301-306.

Ellerton, N.F. (1980). *An extension of Piaget's theory of cognitive development*. Research Report, Victoria University of Wellington, N.Z.

Ellerton, N.F. (1986a). Children's made-up mathematics problems - a new perspective on talented mathematicians. *Educational Studies in Mathematics 17*, 261-271.

Ellerton, N.F. (1986b). Mathematics problems written by children. *Research in Mathematics Education in Australia (December)*, 32-44.

Geeslin, W.E. (1977). Using writing about mathematics as a teaching technique. *Arithmetic Teacher 70*, 112-115.

Kennedy, W. (1985). Writing letters to learn math. *Learning (Feb.)*, 59-60.

Labinowicz, E. (1985). *Learning From Children*. Menlo Park (Cal.): Addison Wesley Publishing Co.

McLeod, D.B. (1987a). A constructivist approach to research on attitude toward mathematics. In J.C. Bergeron, N. Herscovics, C. Kieran (Eds.), *Proceedings of the Eleventh International Conference for the Psychology of Mathematics Education*, Montreal: Vol. 1, 133-139.

McLeod, D.B.(1987b). Beliefs, attitudes and emotions: Affective factors in mathematics learning. In J.C. Bergeron, N. Herscovics, C. Kieran (Eds.), *Proceedings of the Eleventh International Conference for the Psychology of Mathematics Education*, Montreal: Vol. 1, 170-180.

Mandler, G. (1984). *Mind and Body: Psychology of emotion and stress*. New York: Norton.

Mett, C.L. (1987). Writing as a learning device in calculus. *Mathematics Teacher 80(7)*, 534-537.

van den Brink, F.J. (1985). Discussion paper presented to the Working Group *Principles of Teaching Design* at the Ninth International Conference, Psychology of Mathematics Education, Noorwijkerhout, The Netherlands, July, 1985.

THE ATTITUDES AND PRACTICES OF STUDENT TEACHERS OF PRIMARY SCHOOL MATHEMATICS

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A group of student primary school teachers were studied with regard to : Knowledge of mathematics (subject specialism), Attitude to mathematics, displayed Confidence and Liking of mathematics teaching (during practice), and Approach to mathematics teaching (open vs. closed). The Mathematics specialists tended to have positive attitudes to mathematics, and to its teaching, but varied in their approach to teaching mathematics: only 40% adopt a creative, problem solving approach. The students with low levels of knowledge of mathematics are more varied in their responses. It seems that attitudes to mathematics are less significant for these students than attitudes to *teaching* mathematics, which (latter) correlate with teaching approach.

It is often claimed that there is an important relationship between the attitudes of teachers, especially to mathematics, and effectiveness of teaching (Bishop and Nickson 1983). The argument is that teacher attitudes influence student attitudes, which have a powerful influence on learning. Indeed a number of researchers have found a significant correlation between teacher attitude and student achievement in mathematics (Begle 1979, Bishop and Nickson 1983, Schofield 1981)

Research has shown that the picture is more complex than this simple argument suggests, for two reasons. First of all, although many researchers have confirmed the existence of a relationship, the correlation between mathematical attitude and achievement is weak (Begle 1979 , Bell et al 1983). The second source of complexity is the multi-dimensional nature of attitude to mathematics. Recent attitude research distinguishes a number of different components of attitude to mathematics as a whole, as well as to specific mathematical topics (Aiken 1976, Schofield and Start 1978, Kulm 1980, Bell et al 1983). In addition, there is also the teacher's attitude to the *teaching* of mathematics. A priori there is every reason to believe that attitude to teaching mathematics may be just as important a factor. On the basis of these considerations a question arises: what is the relationship between attitudes to mathematics and attitudes to its teaching? In addition to interest in the achievement outcomes of mathematics teaching, the Nineteen Eighties has seen an increased concern with the approach or style of mathematics teaching. Official bodies such as the NCTM (1980) in the U.S.A. and the Cockcroft Committee (1982) in the U.K. have strongly recommended the adoption of a creative, problem-solving approach to the mathematics teaching. This raises a second question: to what extent are student teachers' attitudes related to their style of teaching mathematics?

THE STUDY

This is a report of an investigation of a group of student primary school teachers, with regard to their attitudes towards mathematics, and their manifested attitudes and practices in teaching mathematics. In addressing the above questions, the study focusses on the variables:

1. knowledge of mathematics (on the basis of student course specialism)
2. attitude towards mathematics (a combination of their liking of the subject and confidence in their mathematical ability)
3. Liking and enthusiasm for the teaching of mathematics, and
4. Confidence in their ability to teach mathematics (as demonstrated during their first period of practice teaching)
5. Teaching approach in mathematics (creative and exploratory versus narrow and basic computation skills oriented)

The sample consists of 30 students attending a Bachelor of Education Degree course, at an English university. In addition to their primary teaching studies, each student specialises in an academic topic: 10 study Mathematics or Science as their main academic subject (henceforth, the M&S students), 20 study French, History or English (the FH&E students). The sample consists of the 30 students whose supervising tutors on teaching practice cooperated by completing a mathematics teaching observation schedule.

KNOWLEDGE OF MATHEMATICS

The M&S students all have a pass in Mathematics at GCE 'A' level, indicating successful specialist study of mathematics from 16 to 18 years of age, and in addition, they study mathematics in their undergraduate course. None of the FH&E students (with one exception) have a pass in Mathematics at GCE 'A' level, nor do they study mathematics in their undergraduate course. All 30 of the students have passed Mathematics at GCE 'O' level (at age 16) and have taken a Primary Mathematics Curriculum course as undergraduates.

ATTITUDE TO MATHEMATICS

Attitudes were measured by means of a questionnaire, adapted from Dutton (1965), made up of statements concerning liking of and enthusiasm for mathematics, and confidence in mathematics. The overall score is taken to give an undifferentiated measure of 'attitude towards mathematics (as a whole)'. Marks range from 15 to 75 and a score of 40- 50 is taken as indicating a neutral attitude to mathematics. Scores of over 50 (60) are taken as indicating a positive (very positive, respectively) attitude to mathematics.

The questionnaire was administered 4 times during the 18 months of the course, and the overall pattern of attitudes remained more or less constant over the four testings; there was no significant shift in attitudes to mathematics. The test-retest reliability of the last two testings is 0.86. Responses to the questionnaire (fourth testing) are shown in Table 1, below.

TABLE 1 : Student Teachers' Attitudes to Mathematics

STUDENT GROUP	SIZE	MEAN SCORE	S.D.	INTERPRETATION
OVERALL	30	48.2	10.8	NEUTRAL
FH&E	20	44.1	10.2	NEUTRAL
M&S	10	56.6	6.5	POSITIVE

As the large standard deviations in Table 1 suggest, there is a considerable spread in the attitude scores, particularly in the FH&E students. This is shown in Table 2, below.

TABLE 2: The Distribution of Attitudes to Mathematics

ATTITUDE TO MATHEMATICS	ALL (30) STUDENTS	NUMBERS OF M&S STUDENTS	NUMBERS OF FH&E STUDENTS
VERY POSITIVE [>60]	3 (10%)	3 (30%)	0 (0%)
POSITIVE [>50]	12 (40%)	6 (60%)	6 (30%)
NEUTRAL [40 - 50]	7 (23%)	1 (10%)	6 (30%)
NEGATIVE [<40]	6 (20%)	0 (0%)	6 (30%)
VERY NEGATIVE [<30]	2 (7%)	0 (0%)	2 (10%)

Table 2 shows that one half of the 30 students have a positive attitude to mathematics, and that one quarter of the students have a negative attitude to mathematics. In terms of subject groupings: almost all of the M&S students (90%) have a positive attitude to mathematics, and none negative. The picture is quite different for the FH&E students; only 30% of the students have a positive attitude to mathematics, leaving 70% without a positive attitude to mathematics (a subject they will be required to teach to children). Fully 40% of these students have a negative attitude to mathematics

OBSERVATIONS OF THE STUDENTS' TEACHING

During the first teaching practice all of the student teachers taught mathematics for a significant proportion of their time (a mean of 3.5 hours out of 15 hours per week: 23%). Supervising tutors completed a questionnaire on the students' teaching of Mathematics, focussing on a number of factors, including the student teachers':

1. Confidence in teaching mathematics
2. Liking and enthusiasm for the teaching of mathematics (this includes

professed liking, as well as enthusiasm displayed during the teaching of mathematics)

3. Teaching approach in mathematics: creative and exploratory versus narrow and basic computation skills oriented, as evidenced by the use of problem solving and investigation tasks, the use and encouragement of exploratory discussion of mathematical ideas, the degree of concentration on basic computational skills and the teaching that there is a single correct method for each mathematical task.

A summary of the observational data is given in Table 3, below.

TABLE 3: Observational Data on Students' Teaching of Mathematics

	ALL STUDENTS	**M&S STUDENTS**	**FH&E STUDENTS**
CONFIDENCE IN TEACHING MATHS			
Confident in Maths Teaching	20 (67%)	9 (90%)	11 (55%)
Lacking in Confidence	10 (33%)	1 (10%)	9 (45%)
LIKING OF MATHS TEACHING			
Like of Maths Teaching	21 (70%)	8 (80%)	13 (65%)
Neutral	2 (7%)	0 (0%)	2 (10%)
Dislike of Maths Teaching	7 (23%)	2 (20%)	5 (25%)
APPROACH IN TEACHING MATHS			
Creative & Exploratory	8 (27%)	4 (40%)	4 (20%)
Intermediate in Approach	16 (53%)	3 (30%)	13 (65%)
Narrow & Computation Oriented	6 (20%)	3 (30%)	3 (15%)

Table 3 shows that two thirds of all of the student teachers are confident about their teaching of mathematics, and this includes nearly all of the M&S students (90%), but only just over one half of the other students (55%). Almost one half of the non Mathematics and Science specialists (45%) lack confidence in their ability to teach mathematics, an activity which is likely to occupy a large part of their professional life.

Again, about two thirds of all of the students (70%) like, or display enthusiasm whilst teaching mathematics, and this includes most of the M&S students (80%) and about two thirds of the other students (65%). About one quarter of all of the students dislike the teaching of mathematics, and do not display enthusiasm while teaching it, whatever their subject specialism (20% of M&S students and 25% of FH&E students).

With regard to their teaching approach in mathematics, about one quarter (27%) of the students employ a creative and exploratory approach, one fifth (20%) employ a narrow, basic computational skills oriented approach, and one half are intermediate in approach (53%). The proportions of student teachers employing the approaches varies with specialism. Most (65%) of the FH&E students employ an intermediate approach, while most of the M&S students employ one of the more extreme (that is, more strongly characterised) approaches to the teaching of mathematics, namely 70% (M&S) as opposed to 35% (FH&E).

THE RELATIONSHIPS BETWEEN THE VARIABLES

Considering the students with high levels of knowledge of mathematics (the M&S students): they tend to have positive attitudes to mathematics, tend to be confident with regard to their ability to teach mathematics, and tend to like and display enthusiasm during the teaching of mathematics. However, these students do not tend to adopt a creative, exploratory approach to the teaching of mathematics, only a sizable minority (40%) of them do. Another sizable minority (30%) adopt a narrow, basic computation skills approach to the teaching of mathematics. This second grouping does not consist of those M&S students with less positive attitudes, lower levels of confidence in, or liking of teaching mathematics. On the contrary, the students with the most positive attitude to mathematics and the student with the highest level of confidence in teaching mathematics belong to this group. Thus it appears that for some students, the teaching approach adopted is unrelated to the other variables considered.

A conjectured explanation for this involves the students' conceptions of the nature of mathematics. M&S students are likely to view mathematics more as a precise, structured body of truths and methods, or as a more dynamic, creative problem solving activity. Either way, the students can have positive attitudes and a liking and confidence in their teaching of mathematics. What is likely to vary is their teaching approach; students with the former view may tend to approach school mathematics as a rather narrow, computational oriented activity. The latter view may lead to a creative, exploratory approach to the teaching of mathematics (Lerman 1983, Thompson 1984, Ernest 1987). This conjecture accounts for the varying teaching approaches of the M&S students. Needless to say, it remains speculation and requires further empirical investigation (currently in progress).

Concerning students with low levels of knowledge of mathematics (namely the FH&E students): there are no correlations between any two of confidence in teaching mathematics, liking of teaching, and attitude to mathematics. This is perhaps surprising, and is contrary to the experimenter's expectations. With regard to teaching approach, there is a correlation with liking of teaching mathematics (for FH&E students only): a creative, exploratory approach to the teaching of mathematics is correlated to some extent with the students' liking and displayed enthusiasm in the teaching of mathematics. However this correlation is not statistically significant (Pearson's product moment coefficient takes the value 0.32, which is not significant at the 5% level, using the F-test).

It seems that the FH&E students are able to separate attitude to mathematics, and attitude to its teaching, and that other factors than these have more influence on confidence in teaching mathematics. It can be conjectured that liking and enthusiasm for teaching and teaching approach are both influenced by the students' conceptions of the nature of mathematics. A narrow, instrumental view of mathematics as a set of facts and skills is likely to lead to a basic skills oriented approach to teaching mathematics, and an attitude to teaching mathematics as an uninteresting chore. A more creative view of mathematics is more likely to lead to a creative, exploratory approach to teaching, and enthusiasm and liking for the teaching of mathematics.

What this study suggests is that attitude to mathematics may be less important than many previous authors have assumed. Mathematics specialists do have a positive attitude to mathematics, but their knowledge of mathematics is likely to lie behind this, and may be the more important factor. For teachers with lower levels of mathematical knowledge attitudes to the *teaching* of mathematics may be the more important attitudes, as it is these that are associated with a more creative, problem solving approach to mathematics, in the present study. It has also been conjectured that the students' philosophy of mathematics may be an important underlying factor.

REFERENCES

Aiken, L.R. (1970) Review of Educational Research, 40, 551-596.

Begle, E.G. (1979) Critical Variables in Mathematics Education, MAA-NCTM, Washington DC.

Bell, A.W., Costello, J. and Kuchemann, D. (1983) A Review of Research in Mathematical Education, Part A, NFER-Nelson, Windsor.

Bishop, A.J. and Nickson, M. (1983) A Review of Research in Mathematical Education, Part B, NFER-Nelson, Windsor.

Cockroft, W.H. (1982) Mathematics Counts, HMSO, London.

Dutton, W.H. (1965) Journal of Educational Research, 58(8), 362-365.

Ernest, P. (1987), in Preece, P. (ed.) Philosophy and Education (Perspectives 28), School of Education, Exeter University, 88-95.

Kulm, G. (1980) in Shumway, R.J. (ed.) Research in Mathematics Education, NCTM, Reston, Virginia.

Lerman, S. (1983) International Journal of Mathematical Education in Science and Technology, 14(1), 59-66.

NCTM (1980) An Agenda for Action, NCTM, Reston, Virginia.

Schofield, H.L. (1981) Journal of Educational Psychology, 462-471.

Schofield, H.L. and Start, K.B. (1978) The Australian Journal of Education, 22(1), 72-82.

Thompson, A.G. (1984) Educational Studies in Mathematics, 15, 105-127.

CONTEXTS AND PERFORMANCE IN NUMERICAL ACTIVITY AMONG ADULTS

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This is an ongoing report from a study of the ways in which the performance of adults in numerical activity may be related to the context. In scanning the literature to see how the concept of "context", and its relationship to performance, are discussed, I find that a variety of aspects of context are focussed on, yet the way context is specified in empirical work is often somewhat flimsy. Further, many studies which purport to view context as important still report results in a way that suggests that it is something grafted on to a "generic" skill, or neatly separable from an abstractly-conceived task.

This paper is about my attempts to apply an alternative analysis to interviews with adults about a range of numerical problems.

THEORETICAL FRAMEWORK

A literature review of the idea of "context" suggested the following aspects were important for mathematical activity:
material resources e.g. computational technology; goals, beliefs, values; language and "codes"; basis in a social group, social relations; emotional charge (e.g. Maier, 1980; d'Ambrosio, 1985; Cobb, 1986; Carraher and Schliemann, in press).

Yet most research which purports to study different contexts allows only a fairly "one-dimensional" variation. Thus, for example, the U.K. Assessment of Performance Unit (A.P.U.), initially took different "contexts" in the written tests to mean different backgrounds for word problems; e.g., whether a question on ratio was about scoring in a game, or about sharpening numbers of pencils (Foxman et al., 1985, pp.151ff.).

However, the APU, first in the one-to-one practical tests, and from 1987 in group practical tests, has extended its work into other aspects of context mentioned above : language (both written and spoken answers); the material resources (specifying the calculators, etc. available), and social interaction (in the group practical interviews).

Also, ethnographers have aimed to describe in detail the numerate strategies used in work and everyday life (e.g. Lave 1985; Carraher and Schliemann, in press). These studies document the distinctive character of "folk maths" strategies, and their effectiveness, in context. However, their use outside their normal contexts is problematical : thus, for example, the Brazilian street vendors who successfully perform many relevant calculations daily in their heads find "similar" calculations, using pencil and paper, outside the market context, much more difficult, and make many more errors.

In relating performance and context, the fundamental problem is that of deciding whether completing the two types of sum, which would appear the same to a mathematics educator, are "the same task in different contexts", or "different tasks". Of course, in order to make comparisons between performance levels in school and "practical" contexts, as several researchers have done, you must presuppose that the answer to the above dilemma is "the same task in different contexts".

On the other hand, some researchers have challenged this position (e.g. Lave et al., 1984; Walkerdine, 1988). They have insisted that mathematical activity and context cannot be neatly separated : they are mutually influencing, and both are shaped and made meaningful by the larger activity or practice(s) of the subject. The latter are themselves shaped and made meaningful by language. Most practices have some numerate aspects and include procedures for making calculations, measurements, etc.

Practices make available certain "positions" (of power) to people; thus, for example, formal education makes available positions as "teacher" and "pupil". Positions may be different for different social groups - or cultures; thus "going out for dinner" may have a different meaning for men and women.

For a person in a given situation, a particular practice (or occasionally, more than one) is "called up" as relevant for making sense of it; this conditions their conception of the "task", what "skills" they deploy, and the "emotions" experienced.

These ideas led to two research questions for the interviews discussed below :

Question (1) Which practices are called up by the problems posed in the interview ?

Question (2) What are the differences in performance between practical maths and school maths in interview ?

In response, my provisional conjectures were :

(1) There are two main practices with related positions, available in this situation, viz.

(T) College maths, with positions : teacher/student; and

(R) social research with positions : researcher/respondent.

Here, (T) may tend to be called up far more frequently than (R).

(2) To the extent that (R), rather than (T), is called up, students will have more access to "skills" etc. from practices other than school maths. They will experience fewer negative "emotions", and will "perform" better.

METHODOLOGY

The setting was a U.K. Polytechnic with a relatively high proportion of 'mature' students (over 21 years of age, returning to study after some years of work or child-care).

A number (n=25) of interviews were conducted in 1985 and 1986 with Social Science degree students at the end of their 1st year (which includes a maths/stats. course). They were presented with a number of "practical" problems - e.g. reading graphs, deciding how much (if at all) they would tip after a meal, deciding which bottle of tomato sauce they would buy - and were asked three questions for each :

(i) which of their current activities it reminded them of;

(ii) how they were thinking about the question , and their answer; and

(iii) what the problem reminded them of in their early experiences with numbers.

In order to ascertain which practice was called up, I drew on various indicators (Walkerdine, 1988, Ch. 3) :

(A) the explicit discursive features of the task/situation :

* e.g. how the task was introduced, in the interview "script" : as "maths", "test", "research", "views", etc.

(B) "unscripted" aspects of the researcher's performance :

* e.g. different verbal or vocal signs for "correct" and "incorrect" answers;

(C) responses and comments given by the student during the interview :

* the language used in answering the problems;
* especially the response to question (i) above;

(D) reflexive accounts :

*e.g. the ways in which I had been in the position of "maths teacher" to each student.

The indicators of "performance", its "level" and quality, were :

* what the subject said while thinking about the problem;
* how it was said; and
* the apparent "correctness" of the response.

RESULTS

Some findings have already been reported, especially about gender differences in affective response / maths anxiety (Evans, 1987). Here, several "episodes" from one interview will be presented, to show how this methodology can be applied to the interview transcripts, and to illustrate the results. These relate to interview no. 10 - male, middle class (by own occupation) / working class (by parents'), aged in his forties. He had worked in the money markets in London's financial area before joining the course, to qualify in Social Work or Town Planning.

1. As with all students, a number of practices with numerate aspects were called up at different points of the interview; for example :

- providing for a household, by considering whether to install a water meter (in response to a pie-chart showing water use by various sectors of the economy);
- school maths and college maths ;
- several work practices (see below).

What was rather less typical was his ability to think across several practices in the same episode. For example, in response to a graph showing how the price of gold varied in one day's trading in London, students were asked : "Which part of the graph shows where the price was rising fastest? What was the lowest price that day?"

> JE : Does that remind you of anything that you do these days, or you've done recently?
>
> S : Er, some of the work we done in Phase One <the first two terms of the College course>, but if you ask me straight out of my head, what it reminds me of - I worked once with a credit company and we had charts on the wall, trying to galvanise each of us to do better than the other (JE: uh huh), and these soddin' things were always there and we seemed to be slaves to the charts...< 6 lines of transcript>...That's what that reminds of - a bad feeling in a way - I felt that a human being was being judged by that bit of paper....<pp. 8-9 of the transcript>

Here we notice that the student is reminded both of his course - "College maths" - and of his earlier practice of managing a sales team.

2. For many students, negative affect is part of school maths or College maths. For several subjects, including this one in this episode, "bad feeling" is associated with "work maths".

3. In the next episode, the student begins by mentioning college maths, then seems to link work maths with it.

> JE : ...Does it remind you of Phase One?
>
> S : Yeah , well, we done some of the questions like this, and ,er, the RUN over the RISE and that kind of thing...<5 sec.>...trends, I suppose if you were judging a trend ...<2 lines of transcript>...I like the fact I

> can do a chart now (JE : uh huh), but even to do a chart like that now , I couldn't sit down and do it straight away...<3 lines>...With maths I have to go back to the basic things all the time....<pp.10-11>

Here he uses the language of College maths, describing the gradient as "run" over "rise" . He then seems to call up work practices at the same time, given that the terms 'trend' (rather than 'gradient') and 'chart' (rather than 'graph') were not used in the teaching in Phase One.

4. We now need to consider whether this apparent ability to <u>transfer</u> elements from one practice to another will help in performance. In this second episode, he has described the gradient as "run" (X2-X1) over "rise" (Y2-Y1) - whereas it is the inverse! At this stage it is difficult to know whether this is due to a memory slip, or to a more basic misconception. In the next episode, he is asked specific questions about the graph.

> JE : Right , okay, may I ask you which part of the graph shows where the price was rising fastest?
>
> S : If I was to make an instant decision, I'd say that one, but obviously want to make it on a count of the line, wouldn't I? <JE : You'd?...> I'd count a line <JE : Uh huh> as it goes up...<25 sec.>... eleven over six and ten over six, so that one's right - in the first one...
>
> JE ...< 2 lines>... And , um what was the lowest price that day?
>
> S : This one here - five hundred and eighty ...<1 line> ...went higher at the close, for some reason...<p.12>

Here we note that, when he is asked to compare the gradient of two lines, he makes a perfectly accurate "instant decision", presumably drawing on his work experience. However, he feels impelled to "count a line", which I take to mean : calculate the gradient by counting squares on the graph, as in college maths. There he gets the correct answer - confirmed presumably by his earlier work practice decision - though his calculations are approximate, as is his reading of the lowest price. At the end of the last episode, he is back in the "money market" practice, as shown by his speculating about the graph's going "higher at the close, for some reason..." Here it appears that the successful transfer from work practices to college has supported his performance in the latter.

PROVISIONAL CONCLUSIONS

Analysis of the remaining interview transcripts is underway, aimed at critically assessing these developing ideas :

1. We can make a reasonable judgement about what activity / practice a respondent has called up, by using the indicators discussed above. These practices are pervasive in shaping or "constructing" task and context.
2. What practice has been called up will relate not only to the "correctness" of performance, but also to the language and reasoning used with the problem.
3. More confused , less "correct" performance may be observed when school / college maths is called up, not only because of memory failure, "misconceptions" etc., but also because of differences in familiarity, and the emotional charge that is part of a practice. Put another way, familiarity is affective, as well as cognitive. (cf. Evans, 1987).
4. Other practices , and their numerate aspects, are not always positively charged affectively. Therefore, the learning of school maths is not necessarily helped by drawing on practical maths examples (see also Adda, 1986).
5. Nor is the "transfer" of "skills" or concepts from school maths to practical maths - or from non-school practices to school or college maths - at all straighforward, because of differences in language, resources, social relations and emotional associations between different practices. (For further specifics, see Walkerdine, 1988, Ch. 4).

REFERENCES

Adda Josette (1986). "Fight against Academic Failure in mathematics". Pp. 58-61 in Damerow Peter et al. ed. (1986). Mathematics for All. Science and Technology Education : Document Series no. 20. Paris . UNESCO.

Carraher, T. and Schliemann, A. (in press). "Culture arithmetic and mathematical models". Cultural Dynamics.

Cobb Paul(1986). "Contexts, Goals, Beliefs and Learning Mathematics". For the Learning of Mathematics. 6, 2, June. 2-9.

D'Ambrosio U. (1985). "Ethnomathematics and its Place in the History and Pedagogy of Mathematics". For the Learning of Mathematics. 5, 1, Feb. 44-48.

Evans, Jeff. (1987). "Anxiety and performance in practical maths at tertiary level : a report of research in progress". Proceedings of PME-11. Montreal. July.

Foxman Derek et al. (1985). Mathematical Development : a Review of Monitoring in Mathematics, 1978-82, Part 1. Assessment of Performance Unit, Dept. of Education and Science, U.K.

Lave Jean Ed. (1985). "The Social Organisation of Knowledge and Practice : A Symposium". Anthropology and Education Quarterly. 16,3 .171-213.

Lave, Jean, Murtaugh, M. and de la Rocha, O. (1984). "The Dialectic of Arithmetic in Grocery Shopping", in Lave, J. and Rogoff, B., eds. (1984). Everyday Cognition: Its Development in Social Context. Harvard University Press.

Maier, E. (1980). "Folk Mathematics". Mathematics Teaching, No. 93, Dec.

Walkerdine, Valerie (1988). The Mastery of Reason. Methuen.

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PRE-SERVICE TEACHERS CONCEPTIONS OF THE RELATIONSHIPS BETWEEN FUNCTIONS AND EQUATIONS

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ABSTRACT

Knowledge and understanding of the relationships between functions and equations as perceived by prospective secondary mathematics teachers (N=152) in the last phase of their professional education was investigated. They were asked to write a definition of a function, to indicate how functions and equations are related to each other, and to find the number of solutions to a quadratic equation, given a positive value and a negative value of the quadratic expression. The findings suggest that these students hold a limited view of functions as equations only, they do not have a way of making sense of the modern definition and lack the ability to relate solutions of equations (algebraic representation) to values of a corresponding function (in graphic representation).

Teachers' subject-matter content knowledge and pedagogical content knowledge influence, of course, their teaching and therefore their students' learning. Teachers get their professional education from both subject-matter and education courses. But we do not know enough about the knowledge they have and how they use it. This study concentrates on prospective secondary mathematics teachers' knowledge and understanding of one of the most important concepts in mathematics - the concept of FUNCTION.

The relationships between functions and equations are part of the concept of function; here we will report on this aspect of the prospective teachers' knowledge

BACKGROUND OF THE STUDY

We will deal with two factors of the relationships between functions and equation: One has to do with the role of equations in the concept definition and concept image of functions. The other has to do with the relationships between values of a function and solutions to an equation.

a) The role of equations in the definition and image of functions.

The definition of functions has changed during the last two centuries. While an 18th century function was an analytic expression, representing the relation between variables, with its graph having no corners (referred to as Euler's definition), in the 20th century function came to be a subset of the Cartesian product of two sets in which each member of the domain is paired with exactly one element of the range, or a less formal definition: A correspondence between two sets that assigns to each element from the first set exactly one element from the second set (Dirichlet-Bourbaki). 200 years ago functions were equations that described the relation between two variables using algebraic expressions. Today's definition of functions is not so limited. Functions do not have to be graphable, to be representable by equations, and their domain and range may be sets of objects other than numbers. Malik (1980) claims that the necessity of teaching the modern definition of function at school level is not at all obvious. He says that Euler's definition from the 18th century covers all the functions used or required in a calculus course, and up to this stage one never confronts a situation where one has to use the modern definition of function. Malik concludes that since only a particular form of functions is used, the student unconsciously accepts this particular form as the definition. Other studies also show that while students are being taught the modern definition of function, the old one serves as the concept image for these students (Marnyanskii, 1965; Markovitz et al, 1986; Vinner and Dreyfus, in press). Secondary teachers are expected to teach functions according to the modern definition which is used in current texts. What is the concept image of functions for them? Is it the earlier one or is it the modern one? What will they teach their students and why?

b) Values of functions as solutions to equations.

Conceptual knowledge is knowledge that is rich in relationships (Hiebert and Lefevre, 1986). Sometimes it is easier to solve an equation by looking at the corresponding function (if it exists) and relating the solutions to the graphical representation. For example, the solutions of $a_n x^n + \ldots + a_1 x + a_0 = 0$ are the x-intercepts of the graph of the function $f(x) = a_n x^n + \ldots a_1 x + a_0$. Flexibility in moving from one representation to another allows one to see rich relationships and to develop a better conceptual understanding. So, when given a familiar algebraic expression in a problem situation it is desirable for students to be able to make connections between the expression and its corresponding function's graphical representation. How flexible is the teachers' understanding of function?

METHOD

The subjects of this study are prospective secondary mathematics teachers in the last phase of their professional education, they are finishing or have already finished their mathematics methods class. Almost all of them are seniors, a few are juniors or post baccalaurete students. This group was selected so that the description of their knowledge will reflect the knowledge teachers have gained during their college education, but before they start real teaching in the field. The subjects come from eight mid-western universities in the U.S.A. who agree to participate in the study.

Data is being gathered in two phases. At the first phase, an open-ended questionnaire is being administratered to the subjects. Information gathered from a written questionnaire is sufficient for a general description of some facets of the teachers' knowledge, but is limited and sometimes hard to interpret. In order to overcome these difficulties the second phase includes interviews with about 10% of the subjects. By probing, asking subjects to explain what they did and why, asking for their reactions as teachers to students' misconceptions and asking questions which are related to the questionnaire but require more general, longer or more thoughtful responses, a more accurate and detailed picture of the subjects' subject-matter content knowledge and pedagogical content knowledge may be developed.

RESULTS

The report here is based on the first phase which has already completed. 152 subjects have answered the questionnaires.

a) The role of equations in the definition and image of functions.

When asked the following question: *Give a definition of a function* , 33 subjects (out of 146 who answered this question) defined function as an equation, an algebraic expression or a formula.

- "A function is an equation with a one-to-one correspondence between the variables".
- "An equation which satisfies the following requirements..."
- "A function is a numerical expression that ..."

A strict categorization was employed to subjects' responses. Unless the words "equation", "algebraic (numerical) expression", or "formula" were mentioned, responses were not included.

When asked the question: *How are functions and equations related to each other?* , an additional 26 subjects said that functions are equations or that rules for functions are equations (without any additional remarks that some functions may not be representable by equations).

- "All functions can be written as equations, but not all equations are functions"
- "They're the same thing."
- "A function is really an equation."

47% (59 out of the 126 subjects who answered both questions) relate functions only to equations in at least one of their answers.

Table 1 categorizes the responses to the two questions with regard to the relationship between functions and equations.

Table 1 - Categorization and distribution of responses to relationship between functions and equations in the two questions.

Relationship between functions and equations(b)	Defined function as an equation(a)			
	Yes	No	No answer	Total N
Functions are equations	12	26	1	39
Equations are functions	8	26	1	35
Some functions are equations	1	7	0	8
Some equations are functions	3	18	0	21
Other	9	34	3	46
No answer	3	17	1	21
Total N	36	128	6	170(c)

(a) As an equation, an algebraic (numerical) expression or a formula.

(b) We didn't distinguish between responses that considered equations to be functions or only rules of functions, etc.

(c) The total is greater than the number of subjects since some subjects belong to more than one category, i.e., 6 subjects claimed that functions and equations are the same thing, and were put in the first two categories of the question about the relationships between functions and equations.

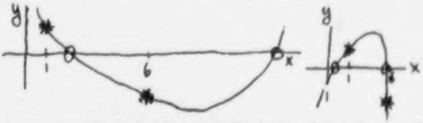
35 subjects responded that equations are functions. At the same time about half of them included in their definition of function the requirement of having one and only one image for each element in the domain. These two contradict each other. Not only that, but

as an answer to another question, a significant number of subjects gave "circle" or "elipse" as examples of graphs of functions.

b) Values of functions as solutions to equations.

When asked the following question: *If you substitute 1 for x in* $ax^2 + bx + c$ *(a, b and c are real numbers), you get a positive number. Substituting 6 gives a negative number. How many real solutions does the equation* $ax^2 + bx + c = 0$ *have? Explain.* 45 subjects (out of 127 who answered this question) gave the correct numerical answer: 2. But only 18 subjects (14%!) got this answer by referring to the graph of a quadratic function and using the Intermediate Value Theorem:

- "Two. The graph is a parabola bv its very nature. If it is positive and crosses the x-axis, it must cross it again."
- "2. This is a parabola. the graph will be either ∪ or ∩ depending on if a is + or −. If x = 1, then y is above the x-axis. If x = 6, y is below the x-axis. It therefore must cross the x-axis. The parabola is symmetric, so there will be 2 x-axis intercepts."



About the same number of subjects (19) claimed that the number of real solutions is two since a polynomial of degree 2 has two real solutions!

- "2 real solutions because it has an order of 2".
- "2 because it is of degree 2"

A large group of subjects (20) just "played" with inequalities without reaching any conclusion.

- "$a + b + c > 0 \quad 36a + 6b + c \leq 0$"
- "$a + b + c > 0 \quad 36a + 6b + c < 0 \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
  the solution is real when $\sqrt{b^2 - 4ac}$ is real"
- "$a + b + c > 0 \quad 36a + 6b + c < 0 \quad a + b + c > 36a + 6b + c$
  $0 > 35b + 5a$
  $-35a > 5b$
  $-7a > b$
  $a < -1/7b$"

Another group considered a, b, c to be variables and reached the conclusion that there are an infinite number of solutions:

- "Since $x = 1$ we have the equation $a + b + c = 0$. There are infinitely many real numbers which satisfy this equation".
- "$(a + b + c) > 0 \quad (36a + 6b + c) < 0 \quad a < 0$.
  (∞) many - dependent on changes in b and c".
- "$a + b + c > 0 \quad 36a + 6b + c < 0$. Infinite, because if $b = c = 1$ then $\exists$ an infinite number values for a that work".

Table 2 shows the distribution of numerical answers and methods of solution used by those who answered the question.

Notice that 8 subjects did use the intermediate value theorem but did not consider the special given function and therefore found only one real solution.

- " 1 real solutions - because it will cross the x-axis in one place."

Table 2 - Distribution of the numerical answers and the methods used to find the number of real solutions.

Method of solution	# of solutions					
	1	2	∞	other	no answer	total N
intermediate value & graph	8	18	—	—	2	28
inequalities or equalities	1	3	4	2	20	30
2nd degree polynomial	1	19	—	—	4	24
a, b, c are variables	—	—	8	1	—	9
other	2	2	12	1	6	23
no explanation	2	3	7	1	—	13
total N	14	45	31	5	32	127

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DISCUSSION

In general more than 20% of the prospective secondary math teachers defined functions as equations. Up to 50% of the subjects showed signs of having equations as the concept image of functions. Even if, as some might claim, this is not important for the functions he/she meets while teaching secondary school, the teacher is expected to teach the modern set definition of function. This might be problematic if it is meaningless to the teacher. It also may contribute to the discrepancy between concept definition and concept image of functions in the next generations of his/her students.

The meaningless of the modern definition to the subjects is also represented by the number of prospective teachers who emphasized the requirement of having one and only one image for each element in the domain but ignored this idea while dealing with functions elsewhere. This might be an explanation for the statement "Equations are functions". Another explanation might be that they ignored some equations when making that statement. In this case it seems that they lack a rich and flexible understanding which would allow them to see a global picture.

A lack of rich relationships, which characterize conceptual knowledge, seems to prevent the prospective teachers from relating the given equation: $ax^2 + bx + c = 0$, to a graphical representation of the function $f(x) = ax^2 + bx + c$. 80% of the subjects did not make the connection. Eisenberg and Dreyfus (1986) report similar findings when in a course which stressed the graphical method of solving inequalities, only 5% of the college students opted for the graphical solution on the exam. Another interesting finding shows that more than 1/4 of the prospective teachers gave answers which don't make sense at all, such as ∞ and even 3, 4 or 5 as the number of solutions of a quadratic equation. Is it because their knowledge is constructed in bits and pieces without connections?

The discussion here is based only on partial results from the questionnaire. These results point to some directions which will be further investigated through interviews. More complete findings will be reported at the meeting.

REFERENCES

Eisenberg, T. and Dreyfus, T. (1986). On visual versus analytical thinking in mathematics. *Proceedings of PME 10*. London, England.

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.) *Conceptual and procedural knowledge: The case of mathematics*. New Jersey: Lawrence Erlbaum Associates, Inc.

Malik, M.A. (1980). Historical and pedagogical aspects of the definition of function. *International Journal of mathematics education in science & technology*, 11.

Markovitz, Z., Eylon, B., & Bruckheimer, M. (1986). Functions today and yesterday, *For the learning of mathematics*, 6 (2), Montreal, Canada.

Marnyanskii, I.A. (1965/1975). Psychological characteristics of pupils' assimilation of the concept of a function. In J. Kilpatrick, I. Wirszup, E. Begle and J. Wilson (Eds.). *Soviet Studies in the psychology of learning and teaching mathematics XIII*. SMSG. USA: University of Chicago Press.

Vinner, S. & Dreyfus, T. (in press). Images and definitions for the concept of function. *Journal of research in mathematics education*.

AN EXPERIMENTAL STUDY OF SOLVING PROBLEMS IN ADDITION AND SUBTRACTION BY FIRST-GRADERS

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ABSTRACT: The experiment held in 3 classes of 2 primary schools stressed the study of the range of the problems in addition and subtraction that should be solved by first-graders, the structure of the relevant teaching materials and the methods of teaching for the development of children's problem-solving and thinking abilities. The results of the experiment indicated that first-graders are able to learn to solve general problems in addition and subtraction if the teaching materials are well-organized according to the inner link of the problems and the teaching stages are appropriately divided according to the characteristics of children's cognition, and the selection of appropriate teaching methods, especially the emphasis on manipulation, the guidance for the children in analysing the information given in the problems an their relations and training them to express their thinking in language greatly promote the development of their problem-solving and thinking abilities.

I. Background and Purpose

There are certain difficulties in teaching low-graders to solve problems in addition and subtraction both at home and abroad. Many children don't get good results in their studies. There are quite a few causes, among which are the problems both in the teaching materials and methods. Studies and experiments on how to teach low-graders to solve problems in addition and subtracton have been carried out both in China and in other countries for the past few years. And there exist different views. For instance,concerning the range and the types of problems in addition and subtraction , some people in China consider it is enough to divide the problems into 2 groups: problems concerning the relation between the sum total and the partial number and problems comparing the difference between 2 numbers. M. Moro of the U.S.S.R. thinks the following three types necessary : problems showing the meaning of addition and subtraction , problems comparing the difference between 2 numbers and problems telling the relation between the known number and the result in addition or subtraction. M. Riley and others of the U.S. hold that problems in addition and subtraction should be divided into the following categories: changing, equalizing , combining and comparing. As to their arrangements, some favour the separate teaching in the first two grades, while others (including M.Moro) maintain the teaching in separate groups in the first grade. As to the teaching methods, some researchers in our country prefer to teach the children the types of problems and their names as well as the formula showing the numerical relation for each type of problems. But there are quite a few who don't agree to such kind of teaching.

For this purpose, the present experiment lay more stress on the study of the follwing questions:

(1) How should the range of the problems in addition and subtraction be decided? Generally speaking, can the first-graders learn to solve these problems?

(2) How can the problems in addition and subtraction be arrranged according to the psychological characteristics of the first-graders so that they will be easy to learn ?

(3) How should children be taught to solve the problems in addition and subtraction and helped to develop their thinking ability?

II. Process of the Experiment and Improvements

The following improvements in the teaching of problems in addition and subtraction have been worked out in our experiment:

1. The problems appeared in the experiment are identical to those appeared in the math teaching materials for low graders with the increase of problems for finding out the subtrahend. And one-step problems in addition and subtraction should be finished learning in Grade 1.

2. The problems are arranged in the following stages:

(1) Solving problems shown in pictures at the beginning of Grade 1. Gradually one of the informations given in a problem is shown in figure instead of clearly drawn out.

(2) Solving problems with both pictures and characters.

(3) Solving word problems, which are divided into three groups, proceeding from the easy to the difficult.

(4) Doing exercises in problems for supplying missing questions or missing data and making-up problems after the learning of each type of problems.

3. The methods for teaching problems in addition and subtraction are improved:

(1) The manipulation with objects by children themselves and demonstration by teachers are strengthened.

(2)The structure of the problems and analysis of numerical relations are stressed.

(3)The ways of solving the problems are closely linked with the meaning of addition and subtraction.

(4)Pay more attention to the guidance and inspiration for children to think.

(5)Give more exercises in comparison and variation.

Our experiment was first held in 3 classes of 2 urban primary schools. The teaching materials were compiled by ourselves. And we prepared the lessons with the teachers taking part in the experiment before each class . We listened to most of the lessons, took notes and looked over the exercises done by the pupils. There was a quiz for each stage. All the notes and the results of each quiz were studied and analysed.

III. Results and Analyses

1. During the 1st term of Grade 1 the transition from problems in pictures to word problems is successfully realized.

2. Our experiment proves that the arrangement of the problems in the teaching materials is conformed to the law for children's cognition--"action, perception--image--concept". When children are taught to solve problems at the beginning, manipulation with objects by themselves should be strengthened, because, compared with mere demonstrations by teachers, it will deepen the children's understanding of the meaning of addition and subtraction and enable them to gain vivid impression on the structure of the problems, make clear their numerical relations and choose the correct way for solution.

CORRECTION RATE FOR 6 PROBLEMS IN 1ST TERM, GRADE 1
(Final Exam)

Problem type	Problems for the sum	Problems for remainder	Problems for 1 addend	Average correction rate
Number of pupils with correct answers(%)	92.7%	91%	92%	91.8%

2. Pupils in the 2nd term of Grade 1 can learn to solve some rather difficult problems in addition and subtraction.

CORRECTION RATE FOR THE PROBLEMS LEARNT IN 2ND TERM, GRADE 1
(Quiz after Learning the problems)

Problem type	Pro. for minuend	Pro. for subtrahend	Pro. for difference between 2 numbers	Pro. for a number large than another	Pro. for a number smaller than another	Average correction rate
Number of pupills with correct answers (%)	90%	89.7 %	94%	97.6% 93.9%*	96.4% 94.9%*	93.8%

* Problems with data described in the opposite way

The experiment shows that while teaching the problems for finding out the minuend or the subtrahend, if we link them with the problems for finding out the remainder, help the children to make clear the relation

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between the known and the unknown through diagrams, they can be understood to be the variations of the problems for finding out the remainder and can be solved by the children applying the meaning of addition and subtraction that they have learned to the new circumstances.

Our experiment also shows that in teaching problems for comparing the difference between 2 numbers, the key point is to let the children know the way to solve the problems for finding out the actual difference between two numbers. They are guided to decide correctly which number is larger (or smaller) through manipulating with objects and using diagrams(as seen below).

○○○○○○ ¦ ○○○
△△△△△△ ¦ 9-6=3

Then they are supposed to find out that the larger number (nine○s) consists of 2 parts: (1) the part similar to the smaller number(six△s) which is 6 and (2) the part larger than the smaller number, which is 3. And if we subtract the part similar to the smaller number from the larger number, we will get the part which is larger than the smaller number.

As to the solving of the problems for finding out the number that is larger or smaller than the known number, we can consider them to be the variations of the problems for finding out the difference between 2 numbers. Children are taught through manipulations to decide which number is larger, analyse the relation between the known and the unknown and find out the way to their solutions.

The above-mentioned problems with data described in the opposite way are taught in the same way. From the results of the experiment, though this kind of problems are more difficult, there is no obvious reduction in the rate of correction in the solution by children.

3. Children's thinking ability is being developed in the process of solving problems in addition and subtraction.

In the solving of problems shown in pictures children are first trained to distinguish between "What does the problem tell?"and "What does the problem ask you to find out?" and then select the correct operation for solving it according to the relation between the above 2 questions. Later exercises for supplying missing questions or missing data are given to the children. Thus their analysing and synthesizing abilities are developed preliminarily. Meanwhile, their reasoning power are cultivated, and their trouble in guessing the solution at random eliminated.

The same train of thought is applied to the solving of problems with the same types of numerical relations, such as the problems for comparing the difference between 2 numbers, and the meaning of addition and subtraction is also referred to. Since children are able to use addition as well as subtraction to solve different types of problems , their transferring ability is also fostered. The flexibility in their thinking is developed through doing exercises of the variations of the problems and the bad habits of memorizing the different types and copying the formulas mechanically are avoided. Moreover, children's ability of reverse thinking is developed through the solution of problems for finding out the minuend or the subtrahend and problems with data described in the opposite way, which are also helpful to the development of the flexibility of their thinking.

IV. Conclusion and Discussion

1.The experiment shows that if the teaching materials are well organized according to the inner link of the problems to be learnt and the principle of proceeding from the easy to the difficult and the concrete to the abstract, and supported by appropriate teaching methods, first-graders can learn to solve general problems in addition and subtraction.

2. The study of the experiment indicates that excellent structure for teaching materials in organizing problems in addition and subtraction will play an important role in helping children to understand and grasp numerical relations and find the ways of solving problems. It is suitable to divide the problems in adddition and subtrraction into the following 3 groups according to their numerical relations:

1st group	Find out the sum of 2 numbers	Sum & one of addends known , find out the other addend	Sum & one of addends known, find out the other addend
2nd group	Find out the minuend	Find out the remainder	Find out the subtrahend
3rd group	For a number larger that the known number (including data described in opposite way)	For difference between 2 numbers	For a number smaller than the known number (including data described in opposite way)

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The underlined one in each group is the prototype, and the other two in the same group are its variants. The teaching by groups will give prominence to the numerical relations among the problems and their mutual connections and be convenient to form cognitive systems with structure in children's minds. When they see any problem in addition or subtraction , they will recognize its characteristics more easily, analyse its numerical relations clearly and choose the way of operation correctly.

3.The study of the experiment also shows that while arranging the teaching materials for the problems in addition and subtraction, we should select teaching methods suited to the psychological characteristics of children. Appropriate teaching methods will promote the successful mastery of the ways to the solution of different problems and the development of children's thinking ability. Owing to the strengthening of pupils' manipulation by using objects and teachers' demonstration, pupils are given more concrete experiences in analysing the numerical relations in different problems and thus enabled to internalize external material activities into intellectual activities, that is, to use their abstract thought for analysing and reasoning. In the process of teaching children to solve problems in addition and subtraction, attention should be paid to the significance of language in the development of their thinking. Asking children questions in order together with manipulation and demonstration will help them to distinguish the information given in the problems from what they ask for, understand the relations between these two and know how to choose the way of operation. All this will urge children to transform their sound language into silent language, namely, to correct thinking.There exists great divergence among children in their understanding of the content and facts of the problems, their analyses of the numerical relations and their choice of the ways of operation. Since the teachers participating in the experiment paid attention to the coaching of backward pupils according to their special needs, especially helping them to learn to think and improve their ability to analyse the problems, better results were achieved. In the quiz given after the teaching of the problems in addition and subtraction, only 2.5 % of the pupils did not pass. And in the final exam there was no failure. This shows that the thinking ability and abilities for solving problems of the backward pupils can be raised so long as the teaching methods adopted are appropriate.

REFERENCE

Ginsburg, H. P. . The Development of Mathematical Thinking. New York, 1983

Moro, M. E. . Methods of Teaching Elementary School Mathematics. Moscow, 1978

Riedesel, C. A.. Teaching Elementary School Mathematics, New Jersey, 1980

BEYOND RATIO FORMULA

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This paper analyses the concept of ratio beyond reasoning on non-integral ratios. Some items for testing the understanding of ratio beyond ratio formula were analysed. Results of the interviews suggested that the level of understanding ratio might be deeper than in previous studies (e.g. Hart, 1981). Errors of a particular geometric task suggested some ideas for developing items for further study.

INTRODUCTION AND METHODOLOGY

In the previous proceedings of PME 1-XI, there are about n ($n > 35$) research reports on ratio and proportion. All those reports dealt with the concept of ratio to the depth of the so-called 'understanding of level four' (Hart, 1981), which means that one must recognize that ratio is needed for the questions, and questions are complex either because they deal with non-integer ratios, such as 5:7, or because of their setting. One who can use the ratio formula ($a:b = c:d$) correctly can always deal successfully with the tasks used by those researchers.

However, beyond the ratio formula, do there still exist developmental hierarchies of understanding ratio? This ongoing research aims to study this problem. Instead of working on searching for the roots of elementary proportional strategies (e.g. Streefland 1984, 1985, and Tourniaier, 1986) this work considered the problem from the other end.

Observing students working in their classrooms, we found that even very able senior high school students, age 16+, were using a familiar relation: $\sin^2 A + \cos^2 A = 1$ to solve the question:

"In $\triangle ABC$, given $\sin A:\sin B:\sin C = 2:3:4$, find $\cos A:\cos B:\cos C = ?$"

As follows: $\sin^2 A:\sin^2 B:\sin^2 C = 4:9:16$,
* $1-\sin^2 A:1-\sin^2 B:1-\sin^2 C = (1-4):(1-9):(1-16) = 3:8:15$
$\cos A:\cos B:\cos C = \sqrt{3}:\sqrt{8}:\sqrt{15}$.

Referred to the above misconceptualised phase *, we developed an interview task:

Given $a:b = 3:4$, find $a - 1 : b - 1 = ?$.

With this task, we had interviewed nine students, age 15+, of understanding level four from one sample school in the previous ratio study (Lin et al, 1985), and found that only one out of nine could cope successfully with this task. This suggested that this task was harder than the items in the previous study (Lin et al, 1985).

Thus, this task, denoted by task A, and a physical task, were used for interviewing at this beginning stage of our study. Five students of understanding level four in a senior high school, aged approximately 17, were identified and interviewed. All of them could cope with task A. However, their responses were at different levels. These two interview tasks and students' responses in the interviews will be analysed in this paper.

Moreover, in order to investigate some errors students might make beyond ratio formula, we will analyse the incorrect responses of a geometric item, which was developed in terms of the task A, in the national entrance test (1986) for further study in senior high school in Taiwan. Out of about 145,000 examinees, age 15+~16+, we randomly sampled 200 copies of the written test papers for this study.

ANALYSIS OF TASKS AND STUDENTS' RESPONSES

1. Task A

Given $a:b = 3:4$, find $a - 1 : b - 1 = ?$

At the syntactic surface structure level (Skemp, 1982) of the symbol system $a:b = 3:4$, we have two letters spatially related to two numbers. At the deep structure level it represents:

(1) two variables a and b
(2) an equivalence class defined by an equivalence relation ($\sim$) in the set of ordered pairs of numbers, formally indicated by $(x, y) \sim (3, 4)$ iff $x:y = 3:4$
(3) $(a, b) \sim (3, 4)$ but it may not necessarily be true that $(a, b) = (3, 4)$
(4) (a, b) depends not only on $(3, 4)$ but also an implicit parameter.

Bearing this deep structure in mind, to respond successfully to the given question $a - 1 : b - 1 = ?$, one needs to realize that

(5) the answer could not be numerical
(6) a parameter has to be included in the answer
(7) the parameter depends completely on a or b.

During the interviews, the question "what does the parameter mean?" was asked. Therefore the five interviewees had to go deeper than just providing a parameter. They were generalizing from their experiences to the meaning of the parameter. With respect to the answer $a - 1 : b - 1 = 3k - 1 : 4k - 1$, three different levels of responses to the question "what is k?" were found.

(i) Two, Lin and Yang, out of five viewed k as the integral greatest common divisor of a and b, and interpreted a and b as two unknowns. Thus k is unique.

(ii) Two, Son and Chou, out of five said k was any non-zero real number, and interpreted a and b as two variables.

(iii) One, Tsai, out of five felt no need to give a closed decision and said "k is a variable; the number system which k belongs to will be determined by variables a and b".

Tsai also said that "If we are only given a:b, we couldn't find $a - 1 : b - 1$. We have to know a:b:1, i.e. a and b needed to be decided". It was surprising in that Tsai mentioned the numerical unit 1, which does not depend on any arbitrariness. With the unit 1, a:b then depends on only one parameter. Thus with the given condition a:b = 3:4, a and b can be completely determined.

2. The Physical Task

"A thermometer showing 40°C was put into a swimming pool full of water at a temperature of 20°C, after 3 minutes the thermometer was 32°C. Try to predict the temperature of this thermometer after 4 minutes and after 9 minutes."

The surface structure of this item is similar to the model of ratio, given three numbers to find the missing value. However, an implicit hypothetical model based on common sense was required to recognize that the model of ratio was not appropriate for this item. The reason for a given time being 9 minutes in the item is to examine how students will reflect while they faced a conflict by a lower temperature of the thermometer than the water if model of ratio was applied.

The model of common sense was very attractive to Tsai, who recognized that ratio was not appropriate for this item without any algorithm.

Tsai: "How can you solve this.... Can't solve it with proportion, because proportion means temperature will be decreased with constant speed. Absolute nonsense. When the difference of temperature is small, the decreasing speed will slow down."

Tsai himself expressed his understanding of dynamic notion of ratio, i.e. moving proportionally means moving with constant speed.

Chou showed his recognition ability after he did his proportional algorithm on 9 minutes. The other three, Lin, Yang and Son, all felt happy about their conclusion that the temperature of the thermometer would reach the same as the water by the time it had decreased to 20°C.

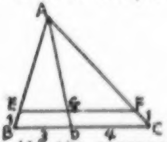
In terms of their recognition ability and interpretations of letters a, b and k in a:b = 3:4, a = 3k and b = 4k, it seems that the responses of (Lin, Yang) and (Chou, Tsai) could be grouped at two levels.

Lin and Yang have not shown any recognition ability on the physical task. They interpreted a and b as generalized numbers and k as the greatest common divisor of a and b.

Chou and Tsai have strong recognition ability. They interpreted a and b as variables and k as a variable within a number system to which a and b belongs. In addition, Tsai's responses provided an index to indicate how deeply a senior high school student could grasp the notion of ratio.

3. The geometric task

Given that the perimeter of $\triangle ABC$ is 21 and $\angle BAD = \angle CAD$.



(1) find AB:AC = ?
(2) find AE:AF = ?
(3) find $\triangle AEG : \triangle AFG$ = ?

In (1) the necessary information BD:CD was given and an implicit theorem to relate AB:AC and BD:CD was required for a satisfactory solution. In (2), the data about AB:AC and AB + AC must be carefully inter-related to produce a satisfactory solution. Two segments BE and CF of length 1 were designed to match the form of task A. Item (2) was our main concern, and the results are as follows:

Out of 200 written test papers sampled for this study, 53 were correct, 76 showed no response, and 71 gave incorrect responses. The percentage of correct responses, 26.5%, of those 200 papers was close to the percentages of correct responses in the population of entrance test data, which was 21%~26% in different areas of Taiwan.

Out of 71 incorrect responses, 7 were non-numerical answers and the rest provided 20 different numerical answers, e.g. 3:4 (17), 2:3 (9), 1:2 (9), 1:1 (4) ... etc. The number in each bracket denotes the frequency. Analysing those incorrect responses, we found that

a. all responses omitted a necessary piece of information, either the perimeter of $\triangle ABC$ or the ratio for sharing with a given sum of two segments.

b. the relation between the incorrect strategies and the numerical answers was not in one-to-one correspondence.

We have classified five different error patterns which would include most of the incorrect responses.

(1) Segments EF and BC do look parallel, so many students applied explicitly or implicitly the incorrect relation, EF parallel to BC, for their solutions. This strategy could provide different numerical answers depending on which implications of parallelism were used, e.g. "AE:AF = BE:FC = 1:1" and
"AE:AF = EG:FG = BD:CD = 3:4".

(2) The algorithm "AB:AC = AE + 1 : AF + 1 * AE:AF = 3:4" indicated that the student was applying an incorrect pattern "c + 1 : d + 1 = c:d. Students' responses might be based on their experience with " $\frac{a}{b} = \frac{c}{d} \leftrightarrow \frac{a}{b} + 1 = \frac{c}{d} + 1$ "

and they may have applied a corollary of the hidden 'theorem': "everything is OK as long as you do the same thing to both sides of an equation" (Lesh et al. 1987).

(3) The algorithm "AB:AC = 3:4, 3 - 1 = 2, 4 - 1 = 3, AE:AF = 2:3" indicated that the student was applying an incorrect pattern "a:b = 3:4 —› a = 3, b = 4". Students were attracted by the surface structure of the symbol system a:b = 3:4. Their attention was drawn away from its deep structure.

(4) Students might derive an equation with two variables from AE + 1 : AF + 1 = 3:4. To solve the equation without using the information AE + AF = 12, they were struggling in performing certain operations to the equation for their answers, e.g. identified two variables into one, tried to eliminate the constant term, etc. The following particular algorithm "AE + 1 : AF + 1 = 3:4, 3AF - 4AE = 1, [illegible] , AE:AF = 2:3" reminded us that ratio reasoning is inter-related to finding integral solutions of a linear equation.

(5) Students might exhibit their competence for finding the sum of two segments, AB + AC = 14 or AE + AF = 12, and then degenerate to a lower level of operation. They intuitively added on some more for a bigger one and subtracted some for a smaller one; e.g.
"21 - (3 + 4 + 2) = 12, AE = 4, AF = 8, AE:AF = 1:2" or
"21 - 7 = 14, 14 ÷ 2 = 7, 7 + 1 = 8 (AC), 7 - 1 = 6 (AB) ..."

Data on the incidence of the above five error patterns have been collected and are being analysed.

REMARKS

The tentative results of the investigation at this stage suggested that these might be hierarchical levels of understanding ratio and proportion beyond level four in the previous studies (Hart, 1981; Lin et al, 1985). This hypothesis requires further investigation.

Referring to error analysis on the geometric task, some ideas for developing items for further study have emerged. We require:

(1) items to show the inter-relation of ratio reasoning and finding integral solutions of linear equation.

(2) items to test students' images or 'sense of' or 'stories' that interpret the operation within a proportion, e.g. changing the middle terms in a proportion.

(3) more recognition items of different understanding levels.

REFERENCES

Hart, K. M., 1981, Children's Understanding of Mathematics: 11-16, London: John Murray.

Lesh, R., Post, T. & Behr, M., 1987, Proportional Reasoning, 12, Paper presented at the Conference on Middle School Number Concepts, Northern Illinois University, May.

Lin, F. L., Kuo, F. P. and Lin, K. H., 1985, Junior High School Children's Understanding of Ratio and Proportion, A second year report of the Mathematics Understanding of Taiwan Students Programme (MUT), Dept. of Mathematics, Taiwan Normal University (in Chinese).

Skemp, R. R., 1982, Communicating Mathematics: Surface Structures and Deep Structures, Visible Language, XVI, 3.

Streefland, L., 1984-85, Search for the Roots of Ratio: Some Thoughts on the Long Term Learning Process (Towards ... a Theory) I. Reflections on a Teaching Experiment. II: The Outline of the Long Term Learning Process, Educational Studies in Mathematics, 15, 4, 327-348; 16, 1, 75-94.

Tourniaier, F., 1986, Proportions in Elementary School, Educational Studies in Mathematics, 17, 4, 401-412.

EYE FIXATIONS DURING THE READING AND SOLUTION OF WORD PROBLEMS CONTAINING EXTRANEOUS INFORMATION: RELATION TO SPATIAL VISUALIZATION ABILITY

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Abstract. The present study utilized eye movement monitoring equipment to investigate the relationship between problem solving behavior and spatial visualization ability in a college sample. Results indicate that spatial visualization ability is positively related to the percentage of time spent viewing information essential to a problem's solution and negatively related to the time spent viewing nonessential material.

INTRODUCTION

Eye movement research has great potential for examining aspects of mathematical problem solving which are not obtainable by traditional means such as verbal protocols and error analysis. Schoenfeld (1985) has urged the use of a variety of methodologies to examine problem solving behavior: "Generally speaking, any methodology...may highlight some aspects of behavior and may obscure or distort others. It is thus prudent to examine particular aspects of problem-solving behavior from as many perspectives as possible, to help separate what resides in the interaction between behavior and methodology" (p. 316). The present study used eye movement monitoring equipment to explore the relationship between spatial visualization ability and the percentage of time spent viewing extraneous and essential information in two-step word problems.

BACKGROUND

Eye Movement Research. The monitoring of eye movements has been widely used in reading research for nearly a century (for reviews, see Just & Carpenter, 1980; Rayner, 1978; Tinker, 1958). Relatively few investigators, however, have employed this technique to study students' approaches to mathematical material such as equations, graphs, word problems, and textbook prose (De Corte & Verschaffel, 1986, 1988; Fry, 1987; Suppes, Cohen, Laddaga, Anliker, & Floyd, 1983; Vonder Embse, 1987).

Eye movement monitoring instruments produce a record of the location and duration of a subject's eye fixations. For relatively uncomplicated mathematical tasks, eye movement registration provides an unobtrusive way to physiologically monitor information processing strategies. Verbal reporting of solution methods is not necessary in this research, although such reports can provide corroborating data on the manner of information processing evidenced by the eye movement record.

Spatial Skills and Mathematical Performance. Driscoll (1983) and Suydam (1980) have provided summaries of characteristics that distinguish successful from less-successful problem solvers. Both reviewers report that successful problem solvers are better able to generalize across problems, identify relevant information, ignore extraneous information, perceive the underlying structure of a problem, and readily identify and implement an orderly plan of attack. Many researchers believe that these characteristics are behavioral manifestations of underlying cognitive styles and abilities. The literature on problem solving is replete with studies correlating problem solving performance (outcomes and strategies) with numerous variables, particularly spatial visualization ability. However, the exact nature of the positive relationship often found between this particular ability and problem solving performance is not clear. This study was designed to test the hypothesis that spatial visualization ability is positively related to the percentage of time spent focusing on essential information in word problems containing nonessential numerical information..

METHODOLOGY

Sample. College undergraduates enrolled in the second of two developmental mathematics courses offered at The Ohio State University served as subjects. The final sample consisted of 26 males and 11 females; the median age of the sample was 20.4, and the mean age was 23.1.

Test of Spatial Visualization Ability. The Mental Rotations Test (Vandenberg, 1971, cited in Vandenberg & Kuse, 1978) was selected as the measure of spatial visualization ability. Subjects were given three minutes to complete each of the two 10-item sections. For each test item, subjects decided which 2 of 4 three-dimensional objects differed from an original in angular orientation only; each choice was denoted with an "X." Figure 1 depicts an item from Part I of this test.

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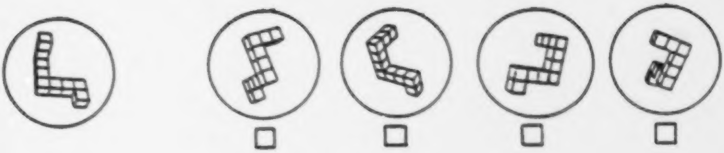


Figure 1. Sample Item from Vandenberg's Mental Rotations Test (MRT)

Eye Movements. Eye movement data were collected using the Micromeasurements System 1200. An IBM PC-XT containing an IBM Data Acquisition and Control Board was responsible for presenting the word problems to the subject and for collecting and formatting the eye position data received from the System 1200 computer. Data on the subject's eye position (horizontal and vertical components) were updated sixty times a second. A fixation was defined as three or more consecutive 1/60-second pupil position readings indicating the same horizontal and vertical positions.

Word Problems. Each subject viewed three 2-step word problems containing extraneous numerical information (see Figures 2-4). Subjects were given 30 seconds to view and try to mentally solve each problem.

ANALYSIS

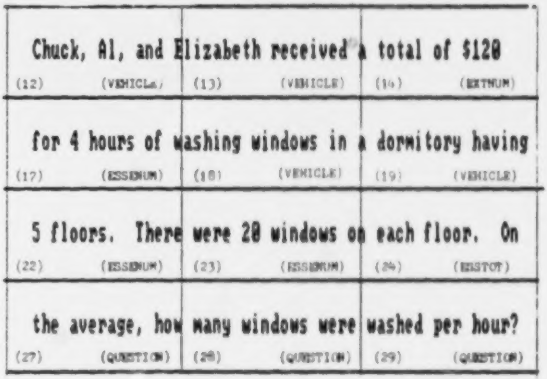
A rectangular grid was superimposed on the text so that the percent of total fixation time for specific regions within each word problem could be analyzed. Horizontal dimensions varied slightly from problem to problem due to the varying locations of features of interest. Grid specifications isolated the following types of information: the essential numerical (ESSENUM), the extraneous numerical (EXTNUM), the essential verbal (which together with ESSENUM formed ESSTOT), the relatively nonessential verbal (VEHICLE), and the question. Figures 2, 3, and 4 show the grid systems and the labels assigned to each block of information for the three word problems. The percent of total fixation time within each block was calculated by dividing the total fixation time within that block by the total fixation time for that problem (30 seconds minus the total saccade time). Scores on the Mental Rotations Test were correlated with each of the five fixation variables described above, as well as with the ratios of EXTNUM to ESSENUM and EXTNUM to ESSTOT.

A fully loade (12) (VEHICLE)	d freight train ha (13) (VEHICLE)	ving 10 cars and (14) (ESSENUM)
an engine wei (17) (ESSTOT)	ghs 1000 tons and (18) (ESSENUM)	is 400 feet long. (19) (EXTNUM)
The engine it (22) (ESSTOT)	self weighs 200 to (23) (ESSENUM)	ns. What is the (24) (QUESTION)
average weigh (27) (QUESTION)	t of each car? (28) (QUESTION)	

Figure 2. Analysis Blocks for Problem 1 and Their Designations

Last week, Gen (14) (VEHICLE)	e worked 6 hour (15) (ESSENUM)	s a day deliveri (16) (ESSTOT)	ng a total of (17) (VEHICLE)
360 telephone (20) (EXTNUM)	books. If he r (21) (VEHICLE)	eceived $120 for (22) (ESSENUM)	5 days' work, (23) (ESSENUM)
how much did h (26) (QUESTION)	e earn per hour? (27) (QUESTION)		

Figure 3. Analysis Blocks for Problem 2 and Their Designations



Chuck, Al, and (12) (VEHICLE)	Elizabeth received (13) (VEHICLE)	a total of $120 (14) (EXTNUM)
for 4 hours of (17) (ESSENUM)	washing windows in (18) (VEHICLE)	a dormitory having (19) (VEHICLE)
5 floors. There (22) (ESSENUM)	were 20 windows on (23) (ESSENUM)	each floor. On (24) (ESSTOT)
the average, how (27) (QUESTION)	many windows were (28) (QUESTION)	washed per hour? (29) (QUESTION)

Figure 4. Analysis Blocks for Problem 3 and Their Designations

RESULTS AND DISCUSSION

Results of the correlational analyses appear in Figure 5. Spatial visualization ability was positively related to the percent of total fixation time on the essential numerical and verbal information in Problem 3 (*r* = .37 for MRT and ESSENUM; *r* = .36 for MRT and ESSTOT). Spatial ability was negatively related to the percent of total fixation time on the extraneous numerical information (*r* = -.56) and to the two ratios described in the Analysis section above (*r* = -.53 for both.) Subjects with high spatial visualization ability spent a greater percentage of time fixating the essential information and less time fixating the extraneous numerical information than did subjects with low spatial ability. Although the correlation coefficients failed to reach significance in Problems 1 and 2, a similar fixation pattern was noted.

Perhaps the placement of the extraneous information in Problem 3 contributed to the robustness of the findings in this particular problem. The extraneous numerical information was placed in the first line of Problem 3 but in the second and third lines of Problems 1 and 2.

Further, in Problem 3, it was the first number mentioned. In Problem 1, it was the third number mentioned, and in Problem 2, it was the second. Subjects with poorly-developed spatial skills may attend to extraneous numerical information for a longer period of time if this information occurs before the essential numbers are mentioned or before the structure of the problem is apparent.

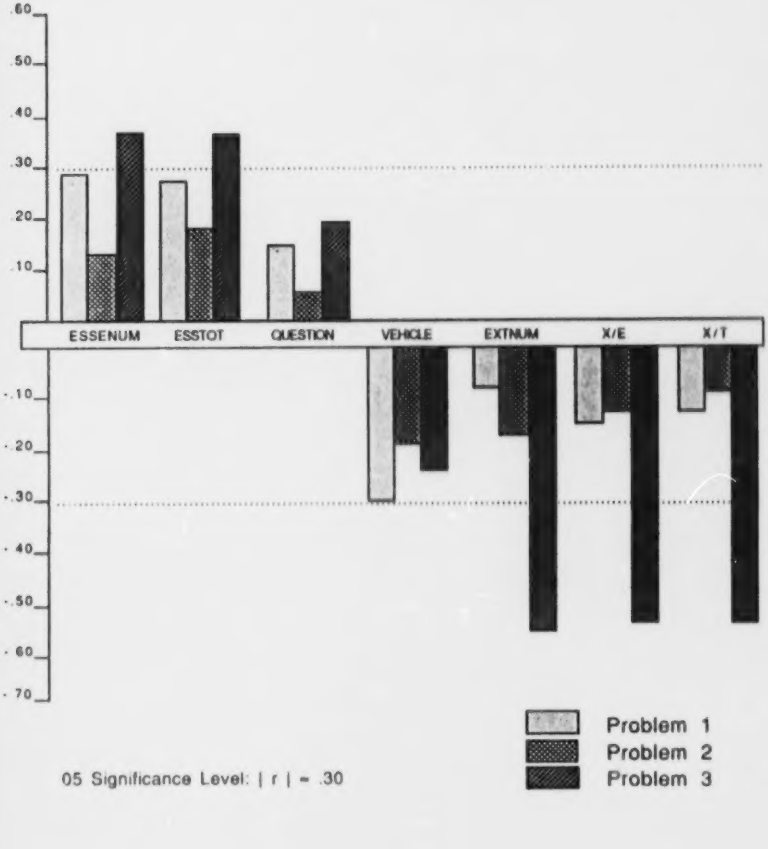


Figure 5. Percent of Total Fixation Time Variables and MRT Scores

CONCLUSION

The findings of this study suggest that students with well-developed spatial skills may perform better than students with poorer spatial skills on mathematical problem solving tasks because the former can more readily identify and attend to the essential information in word problems. To help elucidate any information processing differences that may exist between subjects scoring low on the MRT and those scoring high on this test, an eye scan-path analysis is planned.

The potential of eye tracking research to advance theories of human information processing and to inform mathematics educators of individual differences in problem solving behavior appears considerable. Investigations utilizing eye tracking technology are continuing at Ohio State and are in the planning stages at the University of Illinois (A. Baroody, Personal communication, January 15, 1988). At Ohio State, researchers are examining how undergraduates solve problems in algebra and geometry and what effect question placement may have on eye movement patterns.

REFERENCES

De Corte, E., & Verschaffel, L. (1986). Eye-movement data as access to solution processes of elementary addition and subtraction problems. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco, CA.

De Corte, E., & Verschaffel, L. (1988). First graders' eye movements during elementary addition and subtraction word problem solving. In G. Luer, & U. Lass, (Eds.), Proceedings of the Fourth European Conference on Eye Movements, Volume I (pp. 148-149). Toronto: C. J. Hogrefe.

Driscoll, M. (1983). Research within reach: Secondary school mathematics. Reston, VA: National Council of Teachers of Mathematics.

Fry, C. J. (1987). Eye fixation patterns in the solution of mathematical word problems by young adults: Relation to cognitive style and spatial ability. Unpublished doctoral dissertation, The Ohio State University, Columbus.

Just, M. A., & Carpenter, P. A. (1980). A theory of reading: From eye fixations to comprehension. Psychological Review, 87, 329-354.

Rayner, K. (1978). Eye movements in reading and information processing. Psychological Bulletin, 85, 618-660.

Schoenfeld, A. H. (1985). Mathematical problem solving. Orlando: Academic Press.

Suppes, P., Cohen, M., Laddaga, R., Anliker, J., & Floyd, R. (1983). A procedural theory of eye movements in doing arithmetic. Journal of Mathematical Psychology, 27, 341-369.

Suydam, M. N. (1980). Untangling clues from research on problem solving. In S. Krulik & R. E. Reys (Eds.), Problem solving in school mathematics (1980 NCTM Yearbook) (pp. 34-50). Reston, VA: National Council of Teachers of Mathematics.

Tinker, M. A. (1958). Recent studies of eye movements in reading. Psychological Bulletin, 55, 215-231.

Vandenberg, S. G., & Kuse, A. R. (1978). Mental rotations, a group test of three-dimensional spatial visualization. Perceptual and Motor Skills, 48, 599-604.

Vonder Embse, C. B., Jr. (1987). An eye fixation study of time factors comparing experts and novices when reading and interpreting mathematical graphs. Unpublished doctoral dissertation, The Ohio State University, Columbus.

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THE MEANING OF 'X' IN LINEAR EQUATION AND INEQUALITY : PRELIMINARY SURVEY USING COGNITIVE CONFLICT PROBLEMS

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ABSTRACT

In solving linear equations and inequalities in which the solution set is empty or contains all numbers, the disappearance of x could provoke cognitive conflict in students. By analyzing the students' way of resolving this conflict, this paper aims to clarify the meaning of literal symbol x to junior high school students of Japan. The findings are as follows: 〔1〕 The disappearance of x makes some students try to incorporate x in the final expression or conclude that there is "no answer". These students presuppose the existence of x and regard x as an unknown quantity. 〔2〕 Some students pay more attention to the operation itself and the structure of expressions than to finding a concrete number for x. These students try to examine the truth or falsehood of the intermediate expressions. They may regard x in equations and inequalities as a variable (= a place holder). 〔3〕 Some students accept the final expression with no x as an answer on the basis of their belief that to solve equations and inequalities means to get the final result of transforming the expressions. These students consider x to be no more than an object of transforming the expression.

I. INTRODUCTION

To students, the meaning of x refers mainly to the form such as $x=A$ or $x<A$ obtained by solving equations or inequalities. Some, however, understand that x stands for an unknown quantity and presupposes its existence. When confronted with an equation or inequality in which solution set is empty or contains all numbers, cognitive conflict is provoked since x disappears in the final step. By analyzing the students' way of resolving this conflict, this paper aims to examine the meaning of x to the students.

The Japanese students are generally good at solving linear equations and linear inequalities. However, students may only know what to do without knowing why, or as in R.R. Skemp's terminology, "instrumental understanding" (Skemp, R.R. 1976). This study provides an insight into this observation because the meaning of "x" lies behind the students' procedural efficiency in solving equations and inequalities. It should be noted that the confusion of literal symbols with letters of the alphabet does not occur in Japan because literal symbols in algebra differ from the Japanese written language. Thus, the difficulties experienced differ from those found in English speaking nations(Wagner,S. 1983

). Instead, it is the abstract nature of, and the unfamiliarity of Japanese with the literal symbols which is a problem for Japanese. Though the Japanese seem to be good at manipulating them, this may be superficial. How they grasp the meaning of the literal symbol x is therefore worth investigating.

A thorough diagnostic interview and teaching experiment is deemed necessary for complete understanding but this research is limited to the use of cognitive conflict problems.

II. EDUCATIONAL BACKGROUND OF THE SUBJECTS

Owing to the difficulty mentioned above, literal symbols are carefully introduced into the Course of Study, which is prepared by the Ministry of Education. Although details concerning contents and teaching methods are left to the teachers' discretion, the Course of Study is very influential. Textbooks are likewise approved by the Ministry.

(1) Literal symbols in prescribed textbooks

Algebra does not exist as an independent subject in school mathematics in Japan, but algebra is systematically included in various parts of the mathematics curriculum. In 3rd grade of elementary school, the frame word (□) is introduced, and in 4th grade, an additional frame word is introduced. The following is an example taken from a textbook (4th grade):

> A grid of nails spaced 1cm away from each other is enclosed by an 18cm string. State lengths of the breadth and width of the various rectangles formed. If ○cm represents the breadth and △cm represents the width, what is the relationship between ○ and △ ?

In 5th grade, literal symbols such as "x" and "a" are introduced to stand for quantities formerly represented by the frame words.

(2) Linear equation and inequality topics in prescribed textbooks

In 1st year of junior high school, translations of concrete situations expressed in ordinary language to mathematical sentences using letters and vice versa, are emphasized. Calculations of algebraic expressions required for solving linear equations are studied. Students, then, learn to solve linear equations using the attributes of equality.

In 2nd year, students are required to further develop their abilities to find quantitative relationships, and to express such relationships in a formula by using letters. Computations of a simple formula using letters, and the four fundamental operations are emphasized. By using properties of inequality, students learn to solve linear inequalities. (Miwa,T. 1987)

The students surveyed were eighth graders (N=42) at a national junior high school in Tokyo who had just learned to solve linear inequalities according to the prescribed textbook. It should be noted that they had no prior experience with cognitive conflict problems.

III. THE PROBLEMS GIVEN

The following linear equations and linear inequalities problems were given to students to provoke cognitive conflict.

I. Mr. A solved the equation $3(X+1)-4=3X$ as follows:

$$3(X+1)-4=3X$$
$$3X+3-4=3X$$
$$3X-3X=-3+4$$
$$0=1$$

Here Mr.A got into difficulty.

1. Write down your opinion about Mr. A's solution.
2. Write down your way of solving this equation $3(x+1)-4=3X$ and your reasons.

II. Solve the following inequalities. Please show how you worked out the problem and give reasons for using this method.

① $1-2X<2(6-X)$

② $2(X-1)+3>2X+1$

Problem I is an example of a problem with an empty solution set. To show the "disappearance of x" clearly, the part with which "Mr. A got into difficulty" is expressed as "$0=1$." If it were written as "$0X=1$," students would focus on the meaning of dividing by 0 rather than meaning of x.

The inequalities in problem II are examples of a solution set with all numbers and an empty solution set. In ① the number on the right side was increased, because in the pilot study it was found that if the final expression was given as $0<1$, the students would regard the problem to have "no answer" on the basis that no integer exists between 0 and 1. In ② students got the final expression $0>0$. This problem was given to determine how the concept of x is influenced by 0.

IV. RESULTS

(1) Overview Findings

The number of students who made the correct answers are given as follows:

Problem I	Problem II①	Problem II ②
38	11	29

Students' explanations are as follows:

	Prob.I	Prob.II ①	Prob.II②
Final expression is false(true)	16	(2)	11
Intermediate expression is false(true)*	11	(9)	5
X is disappeared	5	0	0
Other	6	0	4
No reason	0	0	9
	38	(11)	29

[Note] • Students do not aim at the most simple equations and inequalities since x disappears in the process of transformation. Therefore, in this paper, the expressions which students finally got will be called the "final expressions" and the previous ones the "intermediate expressions." As for the intermediate expression being false, the explanation as follows : "In the equation $3X-1=3X$, $3X-1$ cannot be equal to 3X because we are subtracting 1 from the left hand 3X. If we substitute 2 for X, the number on the left side is 1 smaller than that on the right side, as in 5=6."

In problems I and II ②, where the solution set is empty, students tried to account for the disappearance of x by the falsehood of the final and intermediate expressions, but they did not explain of the disappearance of x. However, in problem II ①, where the solution set consists of all numbers, the focus was on "what x was" in this inequality, the solution of equations and inequalities, and "what answers were" since the final expression is still true even if x disappeared. Students were then forced to clarify the meaning of x and of solving equations and inequalities. In the following section the students' explanations to problem II① will be considered.

(2) Answers to Problem II ①

The students' answers to this problem can roughly be divided into the following : $X<0$ or $X<11$, there is no answer, $0<11$, and all numbers. The point which students got into trouble with, as in problems I and II②, was that x disappeared in solving the inequality. Cognitive conflict provokes between previous experience, which states that the answer to an inequality never fails to obtain an expression containing "x", and the present situation, where x disappears. Moreover, the logic in concluding " no answer" based on the falsehood of the final and intermediate expressions in problem I and II ② may also cast another difficulty on the students. In this case, students may have a cognitive conflict between the situation where the disappearance of x means the falsehood of the final and intermediate expressions, and the situation where the final and intermediate expressions are still true in spite of the disappearance of x. The students' reasoning reflects two ways of resolving the cognitive conflict produced by the disappearance of x. One is to persistently regard the answer as an expression containing x, and the other is to accept the answer not containing x. The students' answers to this inequality will be classified as follows:

(i) By giving the correct answer (11 persons)
(ii) By giving an answer to containing x (17 persons)
(iii) By giving an answer not containing x (3 persons)
(iv) By giving no answer (4 persons)
(V) Other (7 persons)

(i) By giving the correct answer

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11 students gave the correct answer, and 10 of these students also gave the correct answers to the other two problems. As for the students' reasons, 2 students, focusing on $0X<11$, considered that "0 is always 0 regardless of what number it is multiplied by." The other 9 students focused on the intermediate expression. For instance, student A, citing the expression $-2X<-2X+11$, gave the explanation that "$-2X$ and $-2X$ are equal no matter what number is substituted for x, so that $-2X$ plus 11 is necessarily bigger than $-2X$." He subsequently explained that "since I don't know how to deal with the form $0<\square$, I will give the intelligible form $-2X<-2X+11$ which is easy to understand." From the student's substitution for x, it can be inferred that he regards x as a variable.

(ii) By giving an answer to containing x

These students regard x as the answer to equations and inequalities finally obtained as a result of the transformation. For them, to solve equations and inequalities is to find the value of x, thus they try to incorporate x in the final expression. At superficial level, the answers can be classified into two types:

T_1: $X<1$ or $X<11$ (7 persons)
T_2: "no answer" (10 persons)

Students of T_1 type can be further divided into two: those who answered $X<0$ or $X<11$ on the basis of the final expression $0<11$ (student B), and those who made use of $X<11+X$ (student C).

Student B: $1-2X<12-2X$
$-2X+2X<12-1$
$X<11$
Answer: X<11 or no answer

Student B stated in problem I that "it is appropriate to assume $0X=X$ since 0 is not 1, or no answer" giving the above transformation for the same reason.

Student C: $-2X+2X<12-1$
$-X+2X<11+X$
$X<11+X$

Student C noted that "I think I have to keep x." In problem I he also notes that "since x must be kept in the final expression, I managed to leave x by multiplying and adding $3X-3X-1$."

In T_1 type, there was a student (student D) who tried to leave x by making an intentional mistake in the calculation as follows:

Student D: $1-2X<2(6-X)$
$-2X<2(6-X)-1$
$-2X/6-X<2-1$
$-2X/6+2X/X<1$
$-X/3<-1$
$X>3$

Reason: "Because I realized that in this way I could keep x."

Student D found the value of x in problem I to be $-4/3$ in a similar way, noting that "to solve an equation is to find the value of x, so that $3X-3X=0$ is nonsense." From this we may guess that it was not difficult for him to calculate $3X-3X$. This is because he persistently tried to incorporate x in the final expression, disclosing a "belief" which lies behind the solution of equations and inequalities.

On the other hand, type T_2 students finally concluded that there was no answer. Some students concluded this based on the disappearance of x in both the equation and the inequality (student E), and other students came to this conclusion based on the falsehood of the final expression of the equation and on the disappearance of x in the inequality (student F).

Student E: $1-2X<12-2X$
$-2X+2X<12-1$
$0<11$ -----> no answer
Reason: "X disappears in $-2X+2X$." (To problem I he noted that
"this problem has no answer because x disappears.")
Student F: "The value of x cannot be found because x disappears in
$-2X+2X=0$. Hence no answer." (In the problem I he noted that
"it cannot be that 0 equals 1. 0 is always 0. Hence no answer.")

Thus the students of T_2 type concluded that there was "no answer" since the result of the calculation did not contain x. They are similar to T_1 type in that they believe that the answer to inequalities shoud contain x. That is, both types of students thought that the solution to equations and inequalities is to get a final expression containing x. For them x is the answer to equations and inequalities, necessarily obtained in the form such as $X=a$ or $X<b$.

(iii) By giving an answer not containing x

There were 3 students of this type. Student G considered that solving equations and inequalities is to calculate on the basis of the properties of equations and inequalities, and that the answer (even if it does not contain x) is the final expression.

Student G: $1-2X<12-2X$
$0<11$
Reason:"I found the answer by calculating the value in the parenthesis."

Student H noted in problem II② that an ordinary calculation would give the result on the left part of this paper, but there seems to be no answer because $0>0$ is obviously false. On the other hand, he gave $0<11$ as the answer to the problem II ①. In the case where the final expression was true, he considered it to be the answer, but if it were false, he concluded that there was "no answer".

(iv) By giving no answer

This type refers to students who gave no explanation or, if any, gave the impression of being at a loss as to what to do. Most of the students of this type concluded that there was "no answer" without giving any reason.

(V) Other

Students of this type gave explanations which focused on the division by 0 or concluded that there was "no answer" from $X=0$ because the left side of the final expression contains 0.

V. THE MEANING OF X IN LINEAR EQUATION AND INEQUALITY

In this section the problem of how students grasp the meaning of x in equations and inequalities, based on the above results, will be discussed.

(1) The Unknown Quantity and the Variable

That students regard the x in equations and inequalities as an unknown quantity can be assumed, based on their previous history of learning. In fact it was in the process of solving a problem containing an unknown quantity that the equation first appeared. That is, to solve an equation meant to find the unknown, the existence of which was assumed. Therefore students believe that the relationship between quantities is the issue in expressing the quantified relation in terms of x, and the existence of x is not questioned. For students who presuppose the existence of x, it is predicted that they are likely to get into difficulty with the interpretation of a final expression which does not contain x. Thus the incompatibility with their experiences would produce cognitive conflict. As a result, some students, incorrectly incorporate x into the final expression (T_1 type), and other students conclude that there is "no answer" based on the disappearance of x (T_2 type).

On the other hand, students who seem to regard x as a variable can come up with the correct answer on the basis of the intermediate expressions by interpreting x to take various values, as in the case of student I ((i) type):

Student I: $1-2X<12-2X$
replacing $-2X$ with □
$1+\square<12+\square$

Reason: "Because the sign of the inequality remains the same even if we add the same number to, or subtract it, from both sides of the expression. Any number will do for □, hence the same applies for x."

Student I may take x as a place holder by replacing $-2X$ with □. Note that he focused on the calculation of adding $-2X$ to both sides without paying attention to finding a concrete number for $-2X$ or X. When he considered the intermediate expressions, he paid more attention to the operation itself and the structure of the expressions than to the objects of calculation such as $-2X$, $+2X$, 1, 12. This is why most (i) type students were able to evaluate the intermediate expressions. This indicates that there were some students who understood x as a variable (= a place holder). On the other hand, (ii) type students, who regarded x as an unknown quantity, tried to find a concrete number for x so they gave answers such as $X<1$ or $X<11$.

(2) The Meaning of X as the Answer to Linear Equations and Inequalities

To solve equations and inequalities means to find the set of values of x that guarantees the truth of the equation and inequality. As shown in section Ⅲ, however, (Ⅲ)-type students, who thought that the solution to equations and inequalities can be reduced to mechanical processes, had the tendency to perfunctorily give an answer not containing x, or to conclude there is "no answer" if the final expression was false. For these students, the goal of solving equations and inequalities is not to find the value of x, but to get the final expression by transforming the original equation and inequality. Therefore the character x in equations and inequalities is regarded as a mere element involved in the process of transforming expressions, and in this sense a final expression with no x can serve as the answer.

V. CONCLUSION

〔1〕 Where the solution set is empty or contains all numbers in equations and inequalities, the disappearance of x makes some students try to incorporate x in the final expression or conclude that there is "no answer". These students presuppose the existence of x and regard x as an unknown quantity.

〔2〕 Some students pay more attention to the operation itself and the structure of expressions than to finding a concrete number for x. These students try to examine the truth or falsehood of the intermediate expressions. They may regard x in equations and inequalities as a variable (= a place holder).

〔3〕 Some students accept the final expression with no x as an answer on the basis of their belief that to solve equations and inequalities means to get the final result of transforming the expressions. These students consider x to be no more than an object of transforming the expression.

REFERENCES

Fujii, T. (1986) "The Role of Cognitive Conflict in Understanding Mathematics" Proceedings of International Group for the Psychology of Mathematics Education(PME-XI, Montreal) Vol.Ⅲ pp.141-147

Miwa,T. (1987) "Algebra Teaching in Japanese School Mathematics" Paper presented at Conference on Teaching and Learning of Algebra, held at Athens Georgia, on March 25-28

Skemp.R.R. (1976) "Relational Understanding and Instrumental Understanding" Mathematics Teaching No.77 pp.20-26

Wagner,S. (1983) "What Are These Things Called Variables ?" Mathematics Teacher Vol.76 Number 7 pp.474-479

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